Wildness of polynomial automorphisms in three variables

Shigeru Kuroda

Department of Mathematics and Information Sciences, Tokyo Metropolitan University, 1-1Minami-Osawa, Hachioji, Tokyo 192-0397, Japan

E-mail address: `kuroda@tmu.ac.jp`

2000 *Mathematics Subject Classification.* Primary 14R10; Secondary 13F20.

Partly supported by the Grant-in-Aid for Young Scientists (B) 21740026, The Ministry of Education, Culture, Sports, Science and Technology, Japan.

Contents

Introduction

For each integral domain R, we denote by $R[\mathbf{x}] = R[x_1, \ldots, x_n]$ the polynomial ring in n variables over R, where $\mathbf{x} = \{x_1, \ldots, x_n\}$ is a set of variables and $n \in \mathbf{N}$. For an R-subalgebra A of $R[\mathbf{x}]$, we consider the automorphism group $\mathrm{Aut}(R[\mathbf{x}]/A)$ of the ring $R[\mathbf{x}]$ over A. We say that $\phi \in \mathrm{Aut}(R[\mathbf{x}]/R)$ is *affine* if $\deg \phi(x_i) = 1$ for $i = 1, \ldots, n$, or equivalently

$$\phi(x_i) = \sum_{j=1}^{n} a_{i,j} x_j + b_i$$

for $i = 1, \ldots, n$ for some $(a_{i,j})_{i,j} \in GL(n, R)$ and $b_1, \ldots, b_n \in R$. Here, $\deg f$ denotes the total degree of f for each $f \in R[\mathbf{x}]$. We say that $\phi \in \mathrm{Aut}(R[\mathbf{x}]/R)$ is *elementary* if ϕ belongs to $\mathrm{Aut}(R[\mathbf{x}]/A_i)$ for some $i \in \{1, \ldots, n\}$, where

$$A_i := R[\mathbf{x} \setminus \{x_i\}]. \tag{0.1}$$

If this is the case, then we have $\phi(x_i) = \alpha x_i + f$ for some $\alpha \in R^{\times}$ and $f \in A_i$, since $\mathrm{Aut}(R[\mathbf{x}]/A_i) = \mathrm{Aut}(A_i[x_i]/A_i)$. By $\mathrm{Aff}(R, \mathbf{x})$, $\mathrm{E}(R, \mathbf{x})$, and $\mathrm{T}(R, \mathbf{x})$, we denote the subgroups of $\mathrm{Aut}(R[\mathbf{x}]/R)$ generated by all the affine automorphisms, all the elementary automorphisms, and $\mathrm{Aff}(R, \mathbf{x}) \cup \mathrm{E}(R, \mathbf{x})$, respectively. We caution that $\mathrm{Aff}(R, \mathbf{y})$ (resp. $\mathrm{E}(R, \mathbf{y})$ and $\mathrm{T}(R, \mathbf{y})$) may not be equal to $\mathrm{Aff}(R, \mathbf{x})$ (resp. $\mathrm{E}(R, \mathbf{x})$ and $\mathrm{T}(R, \mathbf{x})$) for another set $\mathbf{y}$ of variables, i.e., a subset of $R[\mathbf{x}]$ with n elements such that $R[\mathbf{y}] = R[\mathbf{x}]$. When R and $\mathbf{x}$ are clear from the context, we sometimes say that $\phi \in \mathrm{Aut}(R[\mathbf{x}]/R)$ is *tame* if ϕ belongs to $\mathrm{T}(R, \mathbf{x})$, and *wild* otherwise.

The following problem is a fundamental problem in polynomial ring theory.

Tame Generators Problem. When is $\mathrm{T}(R, \mathbf{x})$ equal to $\mathrm{Aut}(R[\mathbf{x}]/R)$?

The equality holds when $n = 1$. In fact, every element of $\mathrm{Aut}(R[\mathbf{x}]/R)$ is affine and elementary.

When $n = 2$, it is known that $\mathrm{T}(R, \mathbf{x}) = \mathrm{Aut}(R[\mathbf{x}]/R)$ if and only if R is a field. Here, the "if" part is due to Jung [**12**] in the case where R is a field of characteristic zero, and to van der Kulk [**14**] in the general case. The "only if" part is due to Nagata [**22**, Exercise 1.6 of Part 2].

Throughout this monograph, k denotes an arbitrary field of characteristic zero. In the case of $n = 3$, Shestakov-Umirbaev [**27**] gave a criterion for deciding whether a given element of $\mathrm{Aut}(k[\mathbf{x}]/k)$ belongs to $\mathrm{T}(k, \mathbf{x})$. As a corollary, they proved a statement equivalent to the following theorem ([**27**, Corollary 10]).

Theorem 1 (Shestakov-Umirbaev)**.** *If $n = 3$ and k is a field of characteristic zero, then it holds that*

$$\mathrm{Aut}(k[\mathbf{x}]/k[x_3]) \cap \mathrm{T}(k, \mathbf{x}) = \mathrm{T}(k[x_3], \{x_1, x_2\}).$$

It was previously known that some elements of $\mathrm{Aut}(k[\mathbf{x}]/k[x_3])$ do not belong to $\mathrm{T}(k[x_3], \{x_1, x_2\})$. For example, Nagata defined $\psi \in \mathrm{Aut}(k[\mathbf{x}]/k[x_3])$ by

(0.2)

$$\psi(x_1) = x_1 - 2(x_1x_3 + x_2^2)x_2 - (x_1x_3 + x_2^2)^2x_3, \quad \psi(x_2) = x_2 + (x_1x_3 + x_2^2)x_3,$$

and proved that ψ does not belong to $\mathrm{T}(k[x_3], \{x_1, x_2\})$. He also conjectured that ψ does not belong to $\mathrm{T}(k, \mathbf{x})$ (see Theorem 1.4 and Conjecture 3.1 of Part 2 of [**22**]). By means of Theorem 1, Shestakov and Umirbaev decided that these elements of $\mathrm{Aut}(k[\mathbf{x}]/k[x_3])$ do not belong to $\mathrm{T}(k, \mathbf{x})$. Thereby, Nagata's conjecture was also solved in the affirmative. For this work, Shestakov and Umirbaev were awarded the E. H. Moore Prize for 2007 by the American Mathematical Society. The Tame Generators Problem is not solved in the other cases.

The purpose of this monograph is to study when elements of $\mathrm{Aut}(k[\mathbf{x}]/k)$ do not belong to $\mathrm{T}(k, \mathbf{x})$ in the case of $n = 3$ in more detail. Let ϕ be an element of $\mathrm{Aut}(k[\mathbf{x}]/k)$. If ϕ fixes two variables, then ϕ is elementary, and hence belongs to $\mathrm{T}(k, \mathbf{x})$. Assume that ϕ fixes exactly one variable. Then, ϕ may not belong to $\mathrm{T}(k, \mathbf{x})$ anymore. Thanks to Theorem 1, the study of such automorphisms is reduced to the case of two variables. For this reason, Part 1 of this monograph is devoted to developing a theory of automorphisms in two variables over a domain. As applications, we prove the wildness of elements of $\mathrm{Aut}(k[\mathbf{x}]/k)$ fixing one variable in various situations.

Recently, the author generalized the theory of Shestakov and Umirbaev in [**15**] and [**16**]. This makes it possible to decide efficiently whether a given element of $\mathrm{Aut}(k[\mathbf{x}]/k)$ belongs to $\mathrm{T}(k, \mathbf{x})$. In Part 2 of this monograph, we deal with elements of $\mathrm{Aut}(k[\mathbf{x}]/k)$ which fix no variable. We prove the wildness of such elements of $\mathrm{Aut}(k[\mathbf{x}]/k)$ using the generalized Shestakov-Umirbaev theory.

Our strategy for studying wildness of automorphisms is as follows. For an R-subalgebra A of $R[\mathbf{x}]$, consider the *tame intersection*

$$\mathrm{Aut}(R[\mathbf{x}]/A) \cap \mathrm{T}(R, \mathbf{x})$$

with the automorphism group over A. If an element of $\mathrm{Aut}(R[\mathbf{x}]/A)$ does not belong to this subgroup, then the element does not belong to $\mathrm{T}(R, \mathbf{x})$ by definition. Hence, determining the tame intersection is the same as giving a sufficient condition for wildness of elements of $\mathrm{Aut}(R[\mathbf{x}]/A)$. For example, Theorem 1 describes the structure of the above subgroup for $R = k$, $n = 3$ and $A = k[x_3]$, from which it was decided that some elements of $\mathrm{Aut}(k[\mathbf{x}]/k[x_3])$ do not belong to $\mathrm{T}(k, \mathbf{x})$. We will prove the wildness of a great many automorphisms by studying tame intersections.

The method of tame intersection is particularly effective in the study of so-called exponential automorphisms. We say that $D \in \mathrm{Der}_R R[\mathbf{x}]$ is *locally nilpotent* if $D^l(f) = 0$ holds for some $l \in \mathbf{N}$ for each $f \in R[\mathbf{x}]$. We denote by $\mathrm{LND}_R R[\mathbf{x}]$ the set of all the locally nilpotent derivations of $R[\mathbf{x}]$ over R.

Assume that R is a $\mathbf{Q}$-domain, and D is an element of $\mathrm{LND}_R\, R[\mathbf{x}]$. Then, we can define the *exponential automorphism* $\exp D \in \mathrm{Aut}(R[\mathbf{x}]/R)$ by

$$(\exp D)(f) = \sum_{l \geq 0} \frac{D^l(f)}{l!}$$

for $f \in R[\mathbf{x}]$. Since various automorphisms of interest have this form, it is of great importance to find conditions for wildness of exponential automorphisms. Note that $\ker D$ is an R-subalgebra of $R[\mathbf{x}]$, and $(\exp D)(f) = f$ holds for each $f \in \ker D$. Hence, $\exp D$ always belongs to $\mathrm{Aut}(R[\mathbf{x}]/\ker D)$. Therefore, the study of the wildness of $\exp D$ is reduced to the study of the tame intersection

$$\mathrm{Aut}(R[\mathbf{x}]/\ker D) \cap \mathrm{T}(R, \mathbf{x}).$$

Generalizing the idea of "tame intersection", we also formulate the notion of "W-test polynomial" in Chapter 5 (Definition 1.1). This is an effective new technique for proving the wildness of automorphisms which are not explicitly defined.

Let us list problems and questions to be studied in this monograph. For $f \in R[\mathbf{x}]$ and $D \in \mathrm{Der}_R\, R[\mathbf{x}]$, it is known that the element fD of $\mathrm{Der}_R\, R[\mathbf{x}]$ is locally nilpotent if and only if $D(f) = 0$ and D is locally nilpotent (cf. [**6**, Corollary 1.3.34]). It is also known that, even if $\exp D$ is tame, $\exp fD$ may be wild for some $f \in \ker D$. It is natural to ask whether the converse is true in general .

Question 1. Let D be a nonzero element of $\mathrm{LND}_R\, R[\mathbf{x}]$. Does $\exp D$ always belong to $\mathrm{T}(R, \mathbf{x})$ if $\exp fD$ belongs to $\mathrm{T}(R, \mathbf{x})$ for some $f \in \ker D \setminus \{0\}$?

We say that $D \in \mathrm{Der}_R\, R[\mathbf{x}]$ is *triangular* if $D(x_i)$ belongs to $R[x_1, \ldots, x_{i-1}]$ for $i = 1, \ldots, n$. If this is the case, then D is locally nilpotent. Moreover, we have

$$(\exp D)(x_i) = x_i + f_i$$

for some $f_i \in R[x_1, \ldots, x_{i-1}]$ for $i = 1, \ldots, n$. Hence, $\exp D$ belongs to $\mathrm{E}(R, \mathbf{x})$, and so belongs to $\mathrm{T}(R, \mathbf{x})$. In general, however, $\exp fD$ does not necessarily belong to $\mathrm{T}(R, \mathbf{x})$ for $f \in \ker D \setminus R$. For example, assume that $n = 3$, and define $D \in \mathrm{Der}_k\, k[\mathbf{x}]$ by

$$D(x_1) = -2x_2, \quad D(x_2) = x_3, \quad D(x_3) = 0.$$

Then, D is triangular if x_1 and x_3 are interchanged, and $f = x_1x_3 + x_2^2$ belongs to $\ker D$. Moreover, $\exp fD$ is equal to Nagata's automorphism defined in (0.2).

The following problem arises naturally.

Problem 2. Assume that $D \in \mathrm{LND}_R\, R[\mathbf{x}]$ is triangular. When does $\exp fD$ belong to $\mathrm{T}(R, \mathbf{x})$ for $f \in \ker D \setminus R$?

As mentioned, it is possible that $\mathrm{T}(R, \mathbf{x})$ is not equal to $\mathrm{T}(R, \mathbf{y})$ for another set $\mathbf{y}$ of variables. Namely, $\phi^{-1} \circ \mathrm{T}(R, \mathbf{x}) \circ \phi$ may not be equal to $\mathrm{T}(R, \mathbf{x})$ for $\phi \in \mathrm{Aut}(R[\mathbf{x}]/R)$. A member of a set of variables is called a *coordinate*. More precisely, we call $f \in R[\mathbf{x}]$ a *coordinate* of $R[\mathbf{x}]$ over R if $R[f, f_2, \ldots, f_n] = R[\mathbf{x}]$ for some $f_2, \ldots, f_n \in R[\mathbf{x}]$, or equivalently $f = \phi(x_1)$ for some $\phi \in \mathrm{Aut}(R[\mathbf{x}]/R)$. A coordinate f of $R[\mathbf{x}]$ over R is said to be *tame*

if $f = \phi(x_1)$ for some $\phi \in \mathrm{T}(R, \mathbf{x})$, and *wild* otherwise. We mention that Jie-Tai Yu conjectured that the two coordinates of $k[\mathbf{x}]$ over k given in (0.2) are wild (the strong Nagata conjecture). The conjecture was solved in the affirmative by Umirbaev-Yu [**29**].

For polynomials, we introduce three notions of wildness. Here, when R is just a domain containing $\mathbf{Z}$, we mean by $\exp D$ the automorphism $\exp \bar{D}$ of $\bar{R}[\mathbf{x}] := \mathbf{Q} \otimes_{\mathbf{Z}} R[\mathbf{x}]$ for each $D \in \mathrm{LND}_R\, R[\mathbf{x}]$, where $\bar{D}$ is the natural extension of D to $\bar{R}[\mathbf{x}]$. Since $R[\mathbf{x}]$ is identified with a subring of $\bar{R}[\mathbf{x}]$, it induces an element of $\mathrm{Aut}(R[\mathbf{x}]/R)$ if $(\exp D)(R[\mathbf{x}]) = R[\mathbf{x}]$.

Definition 0.1. Let f be an element of $R[\mathbf{x}]$.
(i) We say that f is *totally wild* if $\mathrm{Aut}(R[\mathbf{x}]/R[f]) \cap \mathrm{T}(R, \mathbf{x}) = \{\mathrm{id}_{R[\mathbf{x}]}\}$.
(ii) We say that f is *quasi-totally wild* if $\mathrm{Aut}(R[\mathbf{x}]/R[f]) \cap \mathrm{T}(R, \mathbf{x})$ is a finite group.
(iii) Assume that R contains $\mathbf{Z}$. We say that f is *exponentially wild* if there does not exist $D \in \mathrm{LND}_R\, R[\mathbf{x}] \setminus \{0\}$ such that $D(f) = 0$ and $\exp D$ induces an element of $\mathrm{T}(R, \mathbf{x})$.

In order to avoid ambiguity, we sometimes call f a totally (resp. quasi-totally, exponentially) wild element of $R[\mathbf{x}]$ over R if f satisfies (i) (resp. (ii), (iii)) of Definition 0.1. If a coordinate f of $R[\mathbf{x}]$ over R is a totally (resp. quasi-totally, exponentially) wild element of $R[\mathbf{x}]$ over R, then we call f a totally (resp. quasi-totally, exponentially) wild coordinate of $R[\mathbf{x}]$ over R.

By definition, totally wild polynomials are quasi-totally wild. We claim that quasi-totally wild coordinates are wild when $n \geq 2$. In fact, if f is a tame coordinate, and $\phi \in \mathrm{T}(R, \mathbf{x})$ is such that $\phi(x_1) = f$, then $\mathrm{Aut}(R[\mathbf{x}]/R[f]) \cap \mathrm{T}(R, \mathbf{x})$ contains infinitely many automorphisms defined for each $l \geq 0$ by $\phi(x_i) \mapsto \phi(x_i)$ for $i \neq 2$, and $\phi(x_2) \mapsto \phi(x_2 + x_1^l)$.

Assume that R contains $\mathbf{Z}$, and D is a nonzero element of $\mathrm{LND}_R\, R[\mathbf{x}]$. Then, we have $(\exp D)^l = \exp lD \neq \mathrm{id}_{\bar{R}[\mathbf{x}]}$ for each $l \geq 1$. Hence, $\exp D$ has an infinite order. If $\exp D$ induces an element of $\mathrm{Aut}(R[\mathbf{x}]/R)$, then it also has an infinite order. Thus, if $f \in R[\mathbf{x}]$ is quasi-totally wild, then $\exp D$ does not induce an element of $\mathrm{Aut}(R[\mathbf{x}]/R[f]) \cap \mathrm{T}(R, \mathbf{x})$. Consequently, there does not exist $D \in \mathrm{LND}_R\, R[\mathbf{x}] \setminus \{0\}$ such that $D(f) = 0$ and $\exp D$ induces an element of $\mathrm{T}(R, \mathbf{x})$. Therefore, we know that quasi-totally wild polynomials are exponentially wild. When $n = 1$, no coordinate is wild or exponentially wild. We claim that exponentially wild coordinates are wild when $n \geq 2$. In fact, if f is a tame coordinate, and $\phi \in \mathrm{T}(R, \mathbf{x})$ is such that $\phi(x_1) = f$, then the locally nilpotent derivation

$$D := \phi \circ \left(\frac{\partial}{\partial x_2} \right) \circ \phi^{-1} \tag{0.3}$$

kills f and $\exp D$ induces an element of $\mathrm{T}(R, \mathbf{x})$.

Problem 3. Find coordinates which are totally (quasi-totally, exponentially) wild.

The existence of such coordinates means the existence of a very large class of wild automorphisms. It is noteworthy that, if f is a totally wild coordinate, and $f_2, \ldots, f_n \in R[\mathbf{x}]$ are such that $k[f, f_2, \ldots, f_n] = R[\mathbf{x}]$, then

the automorphism defined by

$$f \mapsto f, \quad f_2 \mapsto cf_2 \quad \text{and} \quad f_i \mapsto f_i \quad \text{for} \quad i = 3, \ldots, n \tag{0.4}$$

does not belong to $\mathrm{T}(R, \mathbf{x})$ for each $c \in R^\times \setminus \{1\}$. Similarly, if R contains $\mathbf{Z}$, and if $\phi \in \mathrm{Aut}(R[\mathbf{x}]/R)$ is such that $\phi(x_1)$ is exponentially wild, then the exponential automorphism for the locally nilpotent derivation defined as in (0.3) induces an element of $\mathrm{Aut}(R[\mathbf{x}]/R)$ not belonging to $\mathrm{T}(R, \mathbf{x})$.

We say that $D \in \mathrm{Der}_R R[\mathbf{x}]$ is *triangularizable* if $\tau^{-1} \circ D \circ \tau$ is triangular for some $\tau \in \mathrm{Aut}(R[\mathbf{x}]/R)$. If this is the case, then D is locally nilpotent. Since

$$\exp \tau^{-1} \circ D \circ \tau = \tau^{-1} \circ (\exp D) \circ \tau, \tag{0.5}$$

it follows that $\exp D$ belongs to $\mathrm{T}(R, \mathbf{x})$ if τ belongs to $\mathrm{T}(R, \mathbf{x})$. If τ does not belong to $\mathrm{T}(R, \mathbf{x})$, however, $\exp D$ does not belong to $\mathrm{T}(R, \mathbf{x})$ in general.

Due to the following theorem of Rentschler [**25**], every element of $\mathrm{LND}_k\, k[\mathbf{x}]$ is triangularizable when $n = 2$.

Theorem 2 (Rentschler)**.** *Assume that $n = 2$. For each $D \in \mathrm{LND}_k\, k[\mathbf{x}] \setminus \{0\}$, there exist $\tau \in \mathrm{T}(k, \mathbf{x})$ and $f \in k[x_1] \setminus \{0\}$ such that*

$$\tau^{-1} \circ D \circ \tau = f \frac{\partial}{\partial x_2}.$$

Hence, there exits a coordinate g of $k[\mathbf{x}]$ over k such that $\ker D = k[g]$.

When $n \geq 3$, there exist elements of $\mathrm{LND}_k\, k[\mathbf{x}]$ which are not triangularizable due to Bass [**1**] and Popov [**24**]. Combining this result and a result of Smith [**28**], Freudenburg showed that, for each $n \geq 4$, there exists $D \in \mathrm{LND}_k\, k[\mathbf{x}]$ such that D is not triangularizable and $\exp D$ belongs to $\mathrm{T}(k, \mathbf{x})$ (cf. Lemma 3.36 of [**9**] and the remark following it). So he asked the following question (cf. [**9**, Section 5.3.2]).

Question 4 (Freudenburg)**.** Assume that $n = 3$. Let $D \in \mathrm{LND}_k\, k[\mathbf{x}]$ be such that $\exp D$ belongs to $\mathrm{T}(k, \mathbf{x})$. Is D always triangularizable?

We say that $D \in \mathrm{Der}_R R[\mathbf{x}]$ is *affine* if $\deg D(x_i) \leq 1$ for $i = 1, \ldots, n$. If this is the case, then $\deg D^l(x_i) \leq 1$ holds for each $l \geq 0$. Hence, if $D \in \mathrm{LND}_R R[\mathbf{x}]$ is affine, then we have $\deg (\exp D)(x_i) = 1$ for $i = 1, \ldots, n$. Thus, $\exp D$ belongs to $\mathrm{Aff}(R, \mathbf{x})$, and so belongs to $\mathrm{T}(R, \mathbf{x})$.

We say that $D \in \mathrm{Der}_R R[\mathbf{x}]$ is *tamely triangularizable* if $\tau^{-1} \circ D \circ \tau$ is triangular for some $\tau \in \mathrm{T}(R, \mathbf{x})$, and is *tamely affinizable* if $\tau^{-1} \circ D \circ \tau$ is affine for some $\tau \in \mathrm{T}(R, \mathbf{x})$. If $D \in \mathrm{LND}_R R[\mathbf{x}]$ is tamely triangularizable or tamely affinizable, then it is obvious that $\exp D$ belongs to $\mathrm{T}(R, \mathbf{x})$.

We pose the following problem.

Problem 5. Find conditions under which D is tamely triangularizable or tamely affinizable if and only if $\exp D$ belongs to $\mathrm{T}(R, \mathbf{x})$.

There exists a relation between Question 1 and Problem 5 as follows. Let $\mathcal{D}$ be a subset of $\mathrm{LND}_R R[\mathbf{x}]$ such that fD belongs to $\mathcal{D}$ for each $D \in \mathcal{D}$ and $f \in \ker D$. Assume that D is tamely triangularizable or tamely affinizable for every $D \in \mathcal{D}$ such that $\exp D$ belongs to $\mathrm{T}(R, \mathbf{x})$. Then, Question 1 has an affirmative answer for each $D \in \mathcal{D}$. Actually, if $\exp fD$ belongs to

$\mathrm{T}(R,\mathbf{x})$ for $D \in \mathcal{D}$ and $f \in \ker D \setminus \{0\}$, then there exists $\tau \in \mathrm{T}(R,\mathbf{x})$ such that

$$D' := \tau^{-1}(f)(\tau^{-1} \circ D \circ \tau) = \tau^{-1} \circ (fD) \circ \tau$$

is triangular or affine by assumption, since fD belongs to $\mathcal{D}$. If D' is triangular, then $D_0 := \tau^{-1} \circ D \circ \tau$ must be triangular, since $\tau^{-1}(f) \neq 0$. Hence, $\exp D$ belongs to $\mathrm{T}(R,\mathbf{x})$. If D' is affine, then D_0 must be affine, since $\deg D'(x_i) = \deg \tau^{-1}(f) + \deg D_0(x_i)$. Hence, $\exp D$ belongs to $\mathrm{T}(R,\mathbf{x})$.

We can modify Question 4 as follows.

Question 6. Assume that $n = 3$. Let $D \in \mathrm{LND}_k\, k[\mathbf{x}]$ be such that $\exp D$ belongs to $\mathrm{T}(k,\mathbf{x})$. Is D always tamely triangularizable?

The *rank* $\operatorname{rank} D$ of $D \in \mathrm{Der}_k\, k[\mathbf{x}]$ is by definition the minimal number $r \geq 0$ for which there exists $\sigma \in \mathrm{Aut}(k[\mathbf{x}]/k)$ such that $D(\sigma(x_i)) \neq 0$ for $i = 1, \ldots, r$ (cf. [**8**]). For example, if $n = 2$, then we have $\operatorname{rank} D \leq 1$ for each $D \in \mathrm{LND}_k\, k[\mathbf{x}]$ by Theorem 2. We remark that, if $n \geq 2$ and D is triangular, then D always kills a tame coordinate of $k[\mathbf{x}]$ over k. In fact, D kills x_1 if $D(x_1) = 0$, and $D(x_1)x_2 - f$ if $D(x_1) \neq 0$, where $f \in k[x_1]$ is such that $\partial f/\partial x_1 = D(x_2)$. Hence, if $n \geq 2$ and D is triangular, then we have $\operatorname{rank} D \leq n - 1$. Consequently, if $n \geq 2$ and D is triangularizable, then we have $\operatorname{rank} D \leq n - 1$. Freudenburg [**8**] gave the first examples of $D \in \mathrm{LND}_k\, k[\mathbf{x}]$ with $\operatorname{rank} D = n$ for each $n \geq 3$.

In connection with Question 4, we are interested in the following question.

Question 7. Assume that $n \geq 3$. Does there exist $D \in \mathrm{LND}_k\, k[\mathbf{x}]$ such that $\operatorname{rank} D = n$ and $\exp D$ belongs to $\mathrm{T}(k,\mathbf{x})$?

We mention that, when $n \geq 3$, tameness and wildness of $\exp D$ are not previously determined for any $D \in \mathrm{LND}_k\, k[\mathbf{x}]$ with $\operatorname{rank} D = n$.

The study of polynomial automorphisms is closely related to the study of locally nilpotent derivations, so we are interested in locally nilpotent derivations. The following problem is an important problem in polynomial ring theory with little progress.

Problem 8. Describe all the elements of $\mathrm{LND}_k\, k[\mathbf{x}]$.

When $n = 1$, every element of $\mathrm{LND}_k\, k[\mathbf{x}]$ has the form $c(\partial/\partial x_1)$ for some $c \in k$. When $n = 2$, Theorem 2 gives a solution to the problem. Assume that $n = 3$. If $\operatorname{rank} D \leq 2$, then $\sigma^{-1} \circ D \circ \sigma$ belongs to $\mathrm{LND}_{k[x_3]}\, k[\mathbf{x}]$ for some $\sigma \in \mathrm{Aut}(k[\mathbf{x}]/k)$. Since each element of $\mathrm{LND}_{k[x_3]}\, k[\mathbf{x}]$ naturally extends to an element of $\mathrm{LND}_{k(x_3)}\, k(x_3)[x_1, x_2]$, the problem is reduced to the case of $n \leq 2$. When $\operatorname{rank} D = 3$, the problem remains open. Actually, there is no simple way to find $D \in \mathrm{LND}_k\, k[\mathbf{x}]$ with $\operatorname{rank} D = 3$. The problem is also open when $n \geq 4$ and $\operatorname{rank} D \geq 3$.

The outline of this monograph is as follows (see also [**17**] for a summary of an early version of this monograph). Part 1 deals with automorphisms in two variables over a domain as mentioned. In Chapter 1, we give a useful criterion for deciding wildness of elements of $\mathrm{Aut}(R[\mathbf{x}]/R)$. In the study of polynomial automorphisms, the notion of "elementary reduction" is of great importance. As an analogue of this notion, we introduce the notion of "affine

reduction". Note that $\mathrm{Aff}(R, \mathbf{x})$ is contained in $\mathrm{E}(R, \mathbf{x})$ for some kinds of R. For example, if R is a local ring or a Euclidean domain, then every element of $GL(n, R)$ is a product of elementary matrices. Consequently, $\mathrm{Aff}(R, \mathbf{x})$ is contained in $\mathrm{E}(R, \mathbf{x})$. In general, however, $\mathrm{Aff}(R, \mathbf{x})$ is not contained in $\mathrm{E}(R, \mathbf{x})$, and so $\mathrm{T}(R, \mathbf{x})$ is not equal to $\mathrm{E}(R, \mathbf{x})$. To analyze elements of $\mathrm{T}(R, \mathbf{x}) \setminus \mathrm{E}(R, \mathbf{x})$, the notion of "affine reduction" is essential. For a subgroup G of $\mathrm{Aff}(R, \mathbf{x})$, let G^+ be the subgroup of $\mathrm{T}(R, \mathbf{x})$ generated by G and $\mathrm{E}(R, \mathbf{x})$. Then, we can naturally formulate a criterion for deciding whether a given element of $\mathrm{Aut}(R[\mathbf{x}]/R)$ belongs to G^+ using the notions of "affine reduction" and "elementary reduction" (Theorem 2.1). In the case where $G = \mathrm{Aff}(R, \mathbf{x})$, this gives a tameness criterion for elements of $\mathrm{Aut}(R[\mathbf{x}]/R)$.

Chapter 2 is devoted to developing the machinery to be used in Chapters 3 and 4. For $f \in R[\mathbf{x}] \setminus R$, we say that f is *tamely reduced* over R if

$$\deg_{x_1} \tau(f) + \deg_{x_2} \tau(f) \leq \deg_{x_1} f + \deg_{x_2} f$$

holds for every $\tau \in \mathrm{T}(R, \mathbf{x})$. Here, $\deg_{x_i} f$ denotes the degree of f in x_i for each i. We investigate the properties of such a polynomial using the tameness criterion given in Chapter 1. As a consequence, we determine the structure of the tame intersection

$$H(f) := \mathrm{Aut}(R[\mathbf{x}]/R[f]) \cap \mathrm{T}(R, \mathbf{x})$$

in a coarse sense (Theorems 2.4 and 2.7). This result plays key roles in the proofs of various interesting theorems in the following two chapters.

In Chapter 3, we study tameness and wildness of exponential automorphisms. In Section 1, we answer Question 1 in the affirmative when $n = 2$ (Theorem 1.2), and when $n = 3$, $R = k$ and D kills a tame coordinate of $k[\mathbf{x}]$ over k (Theorem 1.3 (ii)). We solve Problem 5 when $n = 2$ (Theorem 1.1). We answer Question 6 in the affirmative when D kills a tame coordinate of $k[\mathbf{x}]$ over k (Theorem 1.3 (i)). In Section 2, we solve Problem 2 in the cases where $n = 2$ (Theorem 2.2), and where $n = 3$ and $R = k$ (Theorem 2.3). As an application, we describe all the wild automorphisms of $k[\mathbf{x}]$ over k of the form $\exp fD$ for some $D \in \mathrm{LND}_k\, k[\mathbf{x}]$ and $f \in \ker D$ in which D is triangular (Proposition 2.4). In Section 3, we study a problem similar to Problem 2 for affine locally nilpotent derivations instead of triangular derivations, and solve this problem for $n = 2$ (Theorem 3.1).

Assume that R contains $\mathbf{Z}$, and let S be an over domain of R, i.e., a domain which contains R as a subring. Consider a coordinate $f \in R[\mathbf{x}]$ of $S[\mathbf{x}]$ over S with $H(f) \neq \{\mathrm{id}_{R[\mathbf{x}]}\}$ and $\deg_{x_1} f \geq \deg_{x_2} f$ which is tamely reduced over R. The main result of Chapter 4 is a classification of such elements of $R[\mathbf{x}]$. If $\deg_{x_2} f = 0$, then f belongs to $R[x_1]$. Since f is a coordinate of $S[\mathbf{x}]$ over S, it follows that f is a linear polynomial in x_1 over R. In the case where $\deg_{x_2} f \geq 1$, we classify such polynomials into five types of polynomials (Definition 1.1, Theorem 1.2). Moreover, we describe the structure of $H(f)$ for the five types of polynomials (Theorem 1.3). As a consequence, we show that f is exponentially wild if and only if f is quasi-totally wild for a coordinate $f \in R[\mathbf{x}]$ of $S[\mathbf{x}]$ over S (Corollary 1.5 (i)). In Section 5, we apply the results to the coordinate $f_i := (\exp hD)(x_i)$ of

$R[\mathbf{x}]$ over R for $i = 1, 2$. Here, R is a $\mathbf{Q}$-domain, and $D \in \mathrm{LND}_R\, R[\mathbf{x}]$ and $h \in \ker D \setminus R$ are such that D is triangular and $\exp hD$ is wild. Then, we completely determine wildness, quasi-totally wildness and totally wildness of f_i (Theorem 5.4), and thereby solving Problem 3 when $n = 2$ and R is a $\mathbf{Q}$-domain.

Part 2 contains highly technical applications of the generalized Shestakov-Umirbaev theory. Throughout, we assume that $n = 3$, and study the wildness of elements of $\mathrm{Aut}(k[\mathbf{x}]/k)$. In Chapter 5, we briefly review the generalized Shestakov-Umirbaev theory, and derive some consequences needed later. In Chapter 6 we prove that some coordinates of $k[\mathbf{x}]$ over k are quasi-totally wild or totally wild (Corollary 1.2). Thus, we solve Problem 3 for $n = 3$ and $R = k$. This is one of the most difficult result in this monograph.

In Chapter 7, we study Question 7. First, we construct large families of elements of $\mathrm{LND}_k\, k[\mathbf{x}]$ (Theorems 1.1 (i) and 1.5 (i)) by generalizing the construction of Freudenburg [**7**]. Then, we check that most of the members of the families have rank three (Theorems 1.3 and 1.6) by using a technique based on "plinth ideal" (Proposition 12.4). Finally, we completely determine the tameness and wildness of the exponential automorphisms by means of "W-test polynomials" (Theorems 1.1 (iii) and 1.5 (ii)). The result is that $\exp D$ is wild whenever $\operatorname{rank} D = 3$. This gives a partial affirmative answer to Question 7. In the last section, toward the solution of Problem 8 for $n = 3$, we discuss how to get more examples of locally nilpotent derivations of rank three.

We conclude this monograph with problems, questions and conjectures.

Part 1

Automorphisms in two variables over a domain

CHAPTER 1

Tame automorphisms over a domain

1. Graded structures

Graded structures on $R[\mathbf{x}]$ play important roles in the study of $\mathrm{Aut}(R[\mathbf{x}]/R)$. Let Γ be a *totally ordered additive group*, i.e., an additive group equipped with a total ordering such that $\alpha \leq \beta$ implies $\alpha + \gamma \leq \beta + \gamma$ for each $\alpha, \beta, \gamma \in \Gamma$. Then, Γ is torsion-free. In this monograph, we assume that a totally ordered additive group is always finitely generated without mentioning it. Then, it follows that Γ is free. Hence, we sometimes regard Γ as a subgroup of the $\mathbf{Q}$-vector space $\mathbf{Q} \otimes_{\mathbf{Z}} \Gamma$.

Let $\mathbf{w} = (w_1, \ldots, w_n)$ be an n-tuple of elements of Γ. We define the $\mathbf{w}$-*weighted grading*

$$R[\mathbf{x}] = \bigoplus_{\gamma \in \Gamma} R[\mathbf{x}]_\gamma$$

by setting $R[\mathbf{x}]_\gamma$ to be the R-submodule of $R[\mathbf{x}]$ generated by the monomials $x_1^{a_1} \cdots x_n^{a_n}$ for $a_1, \ldots, a_n \in \mathbf{Z}_{\geq 0}$ with $\sum_{i=1}^n a_i w_i = \gamma$ for each $\gamma \in \Gamma$. Here, $\mathbf{Z}_{\geq 0}$ denotes the set of nonnegative integers. The set of positive integers will be denoted by $\mathbf{N}$. We say that $f \in R[\mathbf{x}] \setminus \{0\}$ is $\mathbf{w}$-*homogeneous* if f belongs to $R[\mathbf{x}]_\gamma$ for some $\gamma \in \Gamma$. Write $f \in R[\mathbf{x}] \setminus \{0\}$ as $f = \sum_{\gamma \in \Gamma} f_\gamma$, where $f_\gamma \in R[\mathbf{x}]_\gamma$ for each $\gamma \in \Gamma$. Then, we define the $\mathbf{w}$-*degree* of f by

$$\deg_{\mathbf{w}} f = \max\{\gamma \in \Gamma \mid f_\gamma \neq 0\}.$$

We define $f^{\mathbf{w}} = f_\delta$, where $\delta := \deg_{\mathbf{w}} f$. When $f = 0$, we define $f^{\mathbf{w}} = 0$ and $\deg_{\mathbf{w}} f = -\infty$. Here, $-\infty$ is a symbol which is less than any element of Γ. If $\Gamma = \mathbf{Z}$ and $w_i = 1$ for $i = 1, \ldots, n$, then the $\mathbf{w}$-degree of f is the same as the total degree $\deg f$ of f. When $\Gamma = \mathbf{Z}^n$, we denote $\deg_{x_i} f = \deg_{\mathbf{e}_i} f$ for $i = 1, \ldots, n$, where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are the coordinate unit vectors of $\mathbf{R}^n$. Let $f = g/h$ be an element of the field of fractions of $R[\mathbf{x}]$, where $g, h \in R[\mathbf{x}]$ with $g \neq 0$. Then, we define

$$\deg_{\mathbf{w}} f = \deg_{\mathbf{w}} g - \deg_{\mathbf{w}} h \quad \text{and} \quad f^{\mathbf{w}} = \frac{g^{\mathbf{w}}}{h^{\mathbf{w}}}.$$

We note that this definition does not depend on the choice of g and h.

Next, let Ω be the module of differentials of $R[\mathbf{x}]$ over R, and ω an element of the r-th exterior power $\bigwedge^r \Omega$ of the $R[\mathbf{x}]$-module Ω for $r \in \mathbf{N}$. Then, we may uniquely express

$$\omega = \sum_{1 \leq i_1 < \cdots < i_r \leq n} f_{i_1, \ldots, i_r} dx_{i_1} \wedge \cdots \wedge dx_{i_r}, \tag{1.1}$$

where $f_{i_1, \ldots, i_r} \in R[\mathbf{x}]$ for each $i_1, \ldots, i_r$. Here, df denotes the differential of f for each $f \in R[\mathbf{x}]$. We define the $\mathbf{w}$-*degree* of ω by

$$\deg_{\mathbf{w}} \omega = \max\{\deg_{\mathbf{w}} f_{i_1, \ldots, i_r} x_{i_1} \cdots x_{i_r} \mid 1 \leq i_1 < \cdots < i_r \leq n\}.$$

Then, we have

$$\deg_{\mathbf{w}} df = \max\{\deg_{\mathbf{w}}(\partial f/\partial x_i)x_i \mid i = 1, \ldots, n\} \leq \deg_{\mathbf{w}} f \tag{1.2}$$

for each $f \in k[\mathbf{x}]$, since $df = \sum_{i=1}^{n}(\partial f/\partial x_i)dx_i$. Here, we note that the equality holds if f does not belong to R, R contains $\mathbf{Z}$, and $w_i > 0$ for $i = 1, \ldots, n$. For $f \in R[\mathbf{x}]$, $\omega \in \bigwedge^r \Omega$ and $\eta \in \bigwedge^s \Omega$, we have

$$\deg_{\mathbf{w}} f\omega = \deg_{\mathbf{w}} f + \deg_{\mathbf{w}} \omega \quad \text{and} \quad \deg_{\mathbf{w}} \omega \wedge \eta \leq \deg_{\mathbf{w}} \omega + \deg_{\mathbf{w}} \eta.$$

Take $f_1, \ldots, f_r \in R[\mathbf{x}]$ and set $\omega = df_1 \wedge \cdots \wedge df_r$. Then, it follows that

$$\deg_{\mathbf{w}} \omega \leq \sum_{i=1}^{r} \deg_{\mathbf{w}} df_i \leq \sum_{i=1}^{r} \deg_{\mathbf{w}} f_i. \tag{1.3}$$

When $f_1, \ldots, f_r$ are $\mathbf{w}$-homogeneous, we have $\deg_{\mathbf{w}} \omega = \sum_{i=1}^{r} \deg_{\mathbf{w}} f_i$ if and only if $\omega \neq 0$. Indeed, if ω is written as in (1.1), then the $\mathbf{w}$-degree of each monomial appearing in $f_{i_1,\ldots,i_r} x_{i_1} \cdots x_{i_r}$ is equal to $\sum_{i=1}^{r} \deg_{\mathbf{w}} f_i$. Let K be the field of fractions of R. Then, it is well-known that $\omega = 0$ if $f_1, \ldots, f_r$ are algebraically dependent over K (cf. [**19**, Section 26]). Thus, we have $\deg_{\mathbf{w}} \omega < \sum_{i=1}^{r} \deg_{\mathbf{w}} f_i$ if $f_1^{\mathbf{w}}, \ldots, f_r^{\mathbf{w}}$ are algebraically dependent over K.

For an endomorphism ϕ of the R-algebra $R[\mathbf{x}]$, we define an $n \times n$ matrix by

$$J\phi = \left(\frac{\partial \phi(x_i)}{\partial x_j} \right)_{i,j}.$$

Then, we have $d\phi(x_1) \wedge \cdots \wedge d\phi(x_n) = (\det J\phi) dx_1 \wedge \cdots \wedge dx_n$. If ϕ is an element of $\mathrm{Aut}(R[\mathbf{x}]/R)$, then $\det J\phi$ belongs to $R^{\times}$. Hence, we obtain the inequality

$$\begin{aligned} \deg_{\mathbf{w}} \phi := \sum_{i=1}^{n} \deg_{\mathbf{w}} \phi(x_i) &\geq d\phi(x_1) \wedge \cdots \wedge d\phi(x_n) \\ &= \deg_{\mathbf{w}} (\det J\phi) dx_1 \wedge \cdots \wedge dx_n = \sum_{i=1}^{n} w_i =: |\mathbf{w}|. \end{aligned} \tag{1.4}$$

If $\phi(x_1)^{\mathbf{w}}, \ldots, \phi(x_n)^{\mathbf{w}}$ are algebraically dependent over K, then we have $\deg_{\mathbf{w}} \phi > |\mathbf{w}|$ by the discussion above.

For an R-submodule A of $R[\mathbf{x}]$, we denote by $A^{\mathbf{w}}$ the R-submodule of $R[\mathbf{x}]$ generated by $\{f^{\mathbf{w}} \mid f \in A\}$. If A is an R-subalgebra of $R[\mathbf{x}]$, then $A^{\mathbf{w}}$ forms an R-subalgebra of $R[\mathbf{x}]$, since $(fg)^{\mathbf{w}} = f^{\mathbf{w}} g^{\mathbf{w}}$ holds for each $f, g \in R[\mathbf{x}]$. Clearly, $R[f_1^{\mathbf{w}}, \ldots, f_r^{\mathbf{w}}]$ is contained in $R[f_1, \ldots, f_r]^{\mathbf{w}}$ for $f_1, \ldots, f_r \in R[\mathbf{x}]$. We remark that, if $f_1^{\mathbf{w}}, \ldots, f_r^{\mathbf{w}}$ are algebraically independent over K, then we have

$$R[f_1^{\mathbf{w}}, \ldots, f_r^{\mathbf{w}}] = R[f_1, \ldots, f_r]^{\mathbf{w}}.$$

To see this, take any $h = \sum_{i_1,\ldots,i_r} c_{i_1,\ldots,i_r} f_1^{i_1} \cdots f_r^{i_r} \in R[f_1, \ldots, f_r] \setminus \{0\}$, and define μ to be the maximum among $\deg_{\mathbf{w}} f_1^{i_1} \cdots f_r^{i_r}$ for $i_1, \ldots, i_r$ with $c_{i_1,\ldots,i_r} \neq 0$, and h' to be the sum of

$$(c_{i_1,\ldots,i_r} f_1^{i_1} \cdots f_r^{i_r})^{\mathbf{w}} = c_{i_1,\ldots,i_r} (f_1^{\mathbf{w}})^{i_1} \cdots (f_r^{\mathbf{w}})^{i_r}$$

for $i_1, \ldots, i_r$ such that $\deg_{\mathbf{w}} f_1^{i_1} \cdots f_r^{i_r} = \mu$. Then, h' belongs to $R[\mathbf{x}]_{\mu}$, and is nonzero by the assumption that $f_1^{\mathbf{w}}, \ldots, f_r^{\mathbf{w}}$ are algebraically independent

over K. Hence, we know that $h^{\mathbf{w}} = h'$. Since h' belongs to $R[f_1^{\mathbf{w}}, \dots, f_r^{\mathbf{w}}]$, it follows that so does $h^{\mathbf{w}}$.

Lemma 1.1. *For any* $\mathbf{w} \in \Gamma^n$ *and* $\phi \in \mathrm{Aut}(R[\mathbf{x}]/R)$*, we have*

$$\deg_{\mathbf{w}} \phi \geq |\mathbf{w}|.$$

Furthermore, it holds that $\deg_{\mathbf{w}} \phi = |\mathbf{w}|$ *if and only if* $\phi(x_1)^{\mathbf{w}}, \dots, \phi(x_n)^{\mathbf{w}}$ *are algebraically independent over* K.

Proof. Thanks to (1.4) and the note following it, it suffices to check the "if" part of the last statement. Assume that $\phi(x_1)^{\mathbf{w}}, \dots, \phi(x_n)^{\mathbf{w}}$ are algebraically independent over K. Then, it holds that

$$R[\phi(x_1)^{\mathbf{w}}, \dots, \phi(x_n)^{\mathbf{w}}] = R[\phi(x_1), \dots, \phi(x_n)]^{\mathbf{w}} = R[\mathbf{x}]^{\mathbf{w}} = R[\mathbf{x}]$$

as remarked. Hence, we may define $\psi \in \mathrm{Aut}(R[\mathbf{x}]/R)$ by $\psi(x_i) = \phi(x_i)^{\mathbf{w}}$ for $i = 1, \dots, n$. Then, we have

$$\omega := d\phi(x_1)^{\mathbf{w}} \wedge \cdots \wedge d\phi(x_n)^{\mathbf{w}} = (\det J\psi) dx_1 \wedge \cdots \wedge dx_n.$$

Since $\det J\psi$ belongs to $R^{\times}$, we get $\deg_{\mathbf{w}} \omega = |\mathbf{w}|$. On the other hand, we obtain

$$\deg_{\mathbf{w}} \omega = \sum_{i=1}^{n} \deg_{\mathbf{w}} \phi(x_i)^{\mathbf{w}}$$

by the $\mathbf{w}$-homogeneity of $\phi(x_i)^{\mathbf{w}}$'s. Since $\deg_{\mathbf{w}} \phi(x_i)^{\mathbf{w}} = \deg_{\mathbf{w}} \phi(x_i)$ for each i, this is equal to $\deg_{\mathbf{w}} \phi$. Therefore, we conclude that $\deg_{\mathbf{w}} \phi = |\mathbf{w}|$. □

For a permutation $x_{i_1}, \dots, x_{i_n}$ of $x_1, \dots, x_n$, we define

$$J(R; x_{i_1}, \dots, x_{i_n})$$

to be the set of $\phi \in \mathrm{Aut}(R[\mathbf{x}]/R)$ such that $\phi(R[x_{i_1}, \dots, x_{i_l}])$ is contained in $R[x_{i_1}, \dots, x_{i_l}]$ for $l = 1, \dots, n$. Let ϕ be any element of $J(R; x_{i_1}, \dots, x_{i_n})$. Then, ϕ induces an automorphism of $R[x_{i_1}, \dots, x_{i_l}]$ for $l = 1, \dots, n$. By induction on n, it is easy to check that

$$\phi(x_{i_l}) = a_l x_{i_l} + h_l \tag{1.5}$$

for some $a_l \in R^{\times}$ and $h_l \in R[x_{i_1}, \dots, x_{i_{l-1}}]$ for $l = 1, \dots, n$. From this, we see that $J(R; x_{i_1}, \dots, x_{i_n})$ is a subgroup of $\mathrm{E}(R, \mathbf{x})$.

In the rest of this section, we assume that $n = 2$, and investigate the $\mathbf{w}$-degrees of coordinates of $R[\mathbf{x}]$ over R.

Lemma 1.2. *Let f be a coordinate of* $R[\mathbf{x}]$ *over R, and* $\mathbf{w} = (w_1, w_2)$ *an element of* Γ^2*. Then, the following assertions hold*:
(i) *If f belongs to* $R[x_1]$*, then* $\deg_{\mathbf{w}} f$ *is equal to w_1 or zero, and is greater than or equal to w_1.*
(ii) *If f does not belong to* $R[x_i]$ *and* $w_j > 0$ *for* $i, j \in \{1, 2\}$ *with* $i \neq j$*, then we have* $\deg_{\mathbf{w}} f \geq w_j$.

Proof. (i) Let $\phi \in \mathrm{Aut}(R[\mathbf{x}]/R)$ be such that $\phi(x_1) = f$. Then, ϕ belongs to $J(R; x_1, x_2)$, since f belongs to $R[x_1]$ by assumption. Hence, we have $f = a x_1 + b$ for some $a \in R^{\times}$ and $b \in R$. Thus, we get $\deg_{\mathbf{w}} f = \max\{w_1, \deg_{\mathbf{w}} b\}$. Since $\deg_{\mathbf{w}} b$ is equal to zero or $-\infty$, it follows that $\deg_{\mathbf{w}} f$ is equal to w_1 or zero, and is greater than or equal to w_1.

(ii) We show that the monomial x_j^t appears in f for some $t \geq 1$. Supposing the contrary, $f - c$ is divisible by x_i for some $c \in R$. Since $f - c$ is also a coordinate of $R[\mathbf{x}]$ over R, we have $f - c = ax_i$ for some $a \in R^\times$. It follows that $f = ax_i - c$ belongs to $R[x_i]$, a contradiction. Hence, x_j^t appears in f for some $t \geq 1$. Thus, we get $\deg_{\mathbf{w}} f \geq tw_j$. Since $w_j > 0$ by assumption, we have $tw_j \geq w_j$. Therefore, we conclude that $\deg_{\mathbf{w}} f \geq w_j$. $\square$

Let ϕ be an element of $\mathrm{Aut}(R[\mathbf{x}]/R)$. Then, $\phi(x_1)$ or $\phi(x_2)$ does not belong to $R[x_i]$ for $i = 1, 2$. Hence, if $w_1 > 0$ or $w_2 > 0$, then we have

$$\max\{\deg_{\mathbf{w}} \phi(x_1), \deg_{\mathbf{w}} \phi(x_2)\} > 0 \tag{1.6}$$

by Lemma 1.2 (ii).

2. Affine reductions and elementary reductions

For a subgroup G of $\mathrm{Aff}(R, \mathbf{x})$, we define G^+ to be the subgroup of $\mathrm{T}(R, \mathbf{x})$ generated by $G \cup \mathrm{E}(R, \mathbf{x})$. By definition, we have $\mathrm{Aff}(R, \mathbf{x})^+ = \mathrm{T}(R, \mathbf{x})$ and $\{\mathrm{id}_{R[\mathbf{x}]}\}^+ = \mathrm{E}(R, \mathbf{x})$. In this section, we give a criterion for deciding whether a given element of $\mathrm{Aut}(R[\mathbf{x}]/R)$ belongs to G^+ when $n = 2$.

Let ϕ be an element of $\mathrm{Aut}(R[\mathbf{x}]/R)$, and $\mathbf{w}$ an element of Γ^n. We say that ϕ admits a *G-reduction* for the weight $\mathbf{w}$ if there exists $\alpha \in G$ such that

$$\deg_{\mathbf{w}} \phi \circ \alpha < \deg_{\mathbf{w}} \phi.$$

Here, the composition is defined by $(\sigma \circ \tau)(f) = \sigma(\tau(f))$ for each $\sigma, \tau \in \mathrm{Aut}(R[\mathbf{x}]/R)$ and $f \in R[\mathbf{x}]$ as usual. We call a G-reduction an *affine reduction* if $G = \mathrm{Aff}(R, \mathbf{x})$. We say that ϕ admits an *elementary reduction* for the weight $\mathbf{w}$ if there exists $\epsilon \in \mathrm{Aut}(R[\mathbf{x}]/A_i)$ for some $i \in \{1, \ldots, n\}$ such that

$$\deg_{\mathbf{w}} \phi \circ \epsilon < \deg_{\mathbf{w}} \phi,$$

where we define A_i as in (0.1). This condition is equivalent to the condition that $\phi(x_i)^{\mathbf{w}}$ belongs to $R[\{\phi(x_j) \mid j \neq i\}]^{\mathbf{w}}$ for some i. If there is no fear of confusion, we omit to mention the weight $\mathbf{w}$.

We define $\mathrm{E}_0(R, \mathbf{x})$ to be the subgroup of $\mathrm{Aff}(R, \mathbf{x}) \cap \mathrm{E}(R, \mathbf{x})$ generated by

$$\bigcup_{i=1}^{n} \mathrm{Aut}(R[\mathbf{x}]/A_i) \cap \mathrm{Aff}(R, \mathbf{x}),$$

and $\bar{G}$ to be the subgroup of $\mathrm{Aff}(R, \mathbf{x})$ generated by $G \cup \mathrm{E}_0(R, \mathbf{x})$. Clearly, $\bar{G}$ is contained in G^+. If G contains $\mathrm{E}_0(R, \mathbf{x})$, then $\bar{G}$ is equal to G. Hence, we have $\overline{\mathrm{Aff}(R, \mathbf{x})} = \mathrm{Aff}(R, \mathbf{x})$.

Theorem 2.1. *Assume that $n = 2$. Let G be a subgroup of $\mathrm{Aff}(R, \mathbf{x})$, and $\mathbf{w} \in \Gamma^2$ such that $w_1 > 0$ or $w_2 > 0$. If $\deg_{\mathbf{w}} \phi > |\mathbf{w}|$ holds for $\phi \in G^+$, then ϕ admits a $\bar{G}$-reduction or elementary reduction for the weight $\mathbf{w}$.*

We mention that a similar result is known for $G = \mathrm{Aff}(R, \mathbf{x})$ in the case where $\Gamma = \mathbf{Z}$ and $\mathbf{w} = (1, 1)$ (see for example [**10**, Proposition 1]). In the following, we prove Theorem 2.1 using a technique similar to [**27**].

Assume that $n = 2$. We define $\iota \in \mathrm{Aut}(R[\mathbf{x}]/R)$ by

$$\iota(x_1) = x_2 \quad \text{and} \quad \iota(x_2) = x_1.$$

Then, we have $\iota = \iota_1 \circ \iota_2 \circ \iota_3$, where we define $\iota_1, \iota_2, \iota_3 \in \mathrm{Aut}(R[\mathbf{x}]/R)$ by

$$\begin{array}{lll} \iota_1(x_1) = x_1 - x_2 & \iota_2(x_1) = x_1 & \iota_3(x_1) = x_2 - x_1 \\ \iota_1(x_2) = x_2 & \iota_2(x_2) = x_2 + x_1 & \iota_3(x_2) = x_2. \end{array}$$

Hence, ι belongs to $\mathrm{E}_0(R, \mathbf{x})$. For $i, j \in \{1, 2\}$ with $i \neq j$, we define

$$J_{i,j} = J(R; x_i, x_j) \cup J(R; x_i, x_j) \circ \iota.$$

Then, $J_{i,j}$ is equal to the set of $\phi \in \mathrm{Aut}(R[\mathbf{x}]/R)$ such that $\phi(x_1)$ or $\phi(x_2)$ belongs to $R[x_i]$. Since

$$\iota \circ J(R; x_i, x_j) = J(R; x_j, x_i) \circ \iota, \tag{2.1}$$

we get

$$\begin{aligned} \iota \circ J_{i,j} &= \iota \circ J(R; x_i, x_j) \cup \iota \circ J(R; x_i, x_j) \circ \iota \\ &= J(R; x_j, x_i) \circ \iota \cup J(R; x_j, x_i) = J_{j,i}. \end{aligned}$$

Lemma 2.2. *For $i, j \in \{1, 2\}$ with $i \neq j$, we have the following:*
(i) *ψ^{-1} and $\psi^{-1} \circ \tau$ belong to $J := J_{1,2} \cup J_{2,1}$ for each $\psi, \tau \in J_{i,j}$.*
(ii) *$\phi \circ \tau$ belongs to $J_{i,j}$ for each $\phi \in J(R; x_i, x_j) \circ \iota$ and $\tau \in J_{j,i}$.*

Proof. (i) By assumption, ψ belongs to $J(R; x_i, x_j)$ or $J(R; x_i, x_j) \circ \iota$. If ψ belongs to $J(R; x_i, x_j)$, then ψ^{-1} belongs to $J(R; x_i, x_j)$, since $J(R; x_i, x_j)$ is a group. Since τ is an element of $J_{i,j}$ by assumption, it follows that $\psi^{-1} \circ \tau$ belongs to $J_{i,j}$. If ψ belongs to $J(R; x_i, x_j) \circ \iota$, then ψ^{-1} belongs to $\iota \circ J(R; x_i, x_j)$. By (2.1), it follows that ψ^{-1} belongs to $J(R; x_j, x_i) \circ \iota$, and hence belongs to $J_{j,i}$. Since ψ^{-1} and τ belong to $\iota \circ J(R; x_i, x_j)$ and $J_{i,j}$, respectively, we know that $\psi^{-1} \circ \tau$ belongs to $\iota \circ J_{i,j} = J_{j,i}$. Therefore, ψ^{-1} and $\psi^{-1} \circ \tau$ belong to J in either case.

(ii) Since $\phi \circ \iota$ belongs to $J(R; x_i, x_j)$, and $\iota \circ \tau$ belongs to $\iota \circ J_{j,i} = J_{i,j}$, we know that $\phi \circ \tau = (\phi \circ \iota) \circ (\iota \circ \tau)$ belongs to $J_{i,j}$. □

By Lemma 2.2 (i), we see that J is closed under the inverse operation. Since $\bar{G}$ is a group, it follows that $\bar{G} \cup J$ is closed under the inverse operation. Note that J is contained in $\mathrm{E}(R, \mathbf{x})$, since ι and $J(R; x_i, x_j)$'s are contained in $\mathrm{E}(R, \mathbf{x})$. Hence, $\bar{G} \cup J$ is contained in G^+. Since J contains all the elementary automorphisms, and $\bar{G}$ contains G, we know that G^+ is generated by $\bar{G} \cup J$. Because $\bar{G} \cup J$ is closed under the inverse operation, this implies that G^+ is generated by $\bar{G} \cup J$ as a semigroup.

Let ϕ and $\mathbf{w}$ be any elements of $\mathrm{Aut}(R[\mathbf{x}]/R)$ and Γ^2, respectively. Then, we show that ϕ admits an elementary reduction for the weight $\mathbf{w}$ if and only if $\deg_{\mathbf{w}} \phi \circ \sigma < \deg_{\mathbf{w}} \phi$ for some $\sigma \in J$. Since every elementary automorphism of $R[\mathbf{x}]$ belongs to J, the "only if" part is clear. To prove the "if" part, assume that $\deg_{\mathbf{w}} \phi \circ \sigma < \deg_{\mathbf{w}} \phi$ for some $\sigma \in J$. Take $i, j \in \{1, 2\}$ with $i \neq j$ such that σ belongs to $J_{i,j}$. Then, $\sigma(x_1)$ or $\sigma(x_2)$ belongs to $R[x_i]$. Hence, we may write

$$\sigma(x_p) = \alpha x_i - g \quad \text{and} \quad \sigma(x_q) = \beta x_j - h, \tag{2.2}$$

where $p, q \in \{1, 2\}$ with $p \neq q$, $\alpha, \beta \in R^{\times}$, $g \in R$ and $h \in R[x_i]$. Then, we have

$$\deg_{\mathbf{w}} \phi \circ \sigma = \deg_{\mathbf{w}}(\alpha \phi(x_i) - g) + \deg_{\mathbf{w}}(\beta \phi(x_j) - \phi(h)). \tag{2.3}$$

Since (2.3) is less than $\deg_{\mathbf{w}} \phi = \deg_{\mathbf{w}} \phi(x_i) + \deg_{\mathbf{w}} \phi(x_j)$ by assumption, it follows that

$$\deg_{\mathbf{w}}(\alpha\phi(x_i) - g) < \deg_{\mathbf{w}} \phi(x_i) \quad \text{or} \quad \deg_{\mathbf{w}}(\beta\phi(x_j) - \phi(h)) < \deg_{\mathbf{w}} \phi(x_j).$$

This implies that $\alpha\phi(x_i)^{\mathbf{w}} = g$ or $\beta\phi(x_j)^{\mathbf{w}} = \phi(h)^{\mathbf{w}}$. Since $\alpha^{-1}g$ and $\beta^{-1}\phi(h)^{\mathbf{w}}$ belong to $R[\phi(x_j)]^{\mathbf{w}}$ and $R[\phi(x_i)]^{\mathbf{w}}$, respectively, we know that $\phi(x_i)^{\mathbf{w}}$ belongs to $R[\phi(x_j)]^{\mathbf{w}}$, or $\phi(x_j)^{\mathbf{w}}$ belongs to $R[\phi(x_i)]^{\mathbf{w}}$. Therefore, ϕ admits an elementary reduction for the weight $\mathbf{w}$.

Now, we prove Theorem 2.1. We define $G_{\mathbf{w}}$ to be the set of $\phi \in \mathrm{Aut}(R[\mathbf{x}]/R)$ for which there exist $l \in \mathbf{N}$, $\phi_1, \ldots, \phi_{l-1} \in \mathrm{Aut}(R[\mathbf{x}]/R)$ and $\phi_l \in \bar{G} \cup J$ as follows:

(A)

$\phi_1 = \phi$ and $\deg_{\mathbf{w}} \phi_l = |\mathbf{w}|$;

(B)

$\deg_{\mathbf{w}} \phi_{i+1} < \deg_{\mathbf{w}} \phi_i$ and $\phi_{i+1} = \phi_i \circ \tau_i$ for some $\tau_i \in \bar{G} \cup J$ for $1 \leq i < l$.

Assume that $\phi \in G_{\mathbf{w}}$ satisfies $\deg_{\mathbf{w}} \phi > |\mathbf{w}|$. Then, we have $l \geq 2$ in view of (A). Hence, we see from (B) that $\deg_{\mathbf{w}} \phi \circ \tau < \deg_{\mathbf{w}} \phi$ for some $\tau \in \bar{G} \cup J$. Thus, ϕ admits a $\bar{G}$-reduction or elementary reduction. Therefore, it suffices to verify that G^+ is contained in $G_{\mathbf{w}}$.

The following is a key proposition.

Proposition 2.3. *$\phi \circ \tau$ belongs to $G_{\mathbf{w}}$ for each $\phi \in G_{\mathbf{w}}$ and $\tau \in \bar{G} \cup J$.*

Clearly, $\mathrm{id}_{R[\mathbf{x}]}$ belongs to $\bar{G} \cup J$, and satisfies $\deg_{\mathbf{w}} \mathrm{id}_{R[\mathbf{x}]} = |\mathbf{w}|$. Hence, $\mathrm{id}_{R[\mathbf{x}]}$ belongs to $G_{\mathbf{w}}$. Since G^+ is generated by $\bar{G} \cup J$ as a semigroup, we know by Proposition 2.3 that G^+ is contained in $G_{\mathbf{w}}$. Therefore, Theorem 2.1 follows from Proposition 2.3.

We remark that, if there exists $\tau \in \bar{G} \cup J$ such that $\deg_{\mathbf{w}} \psi \circ \tau < \deg_{\mathbf{w}} \psi$ and $\psi \circ \tau$ belongs to $G_{\mathbf{w}}$ for $\psi \in \mathrm{Aut}(R[\mathbf{x}]/R)$, then ψ belongs to $G_{\mathbf{w}}$ by the definition of $G_{\mathbf{w}}$. From this, we see that Proposition 2.3 holds when $\deg_{\mathbf{w}} \phi \circ \tau > \deg_{\mathbf{w}} \phi$. In fact, $(\phi \circ \tau) \circ \tau^{-1} = \phi$ belongs to $G_{\mathbf{w}}$ by assumption, and τ^{-1} belongs to $\bar{G} \cup J$ because $\bar{G} \cup J$ is closed under the inverse operation.

Lemma 2.4. *Let $\phi \in \mathrm{Aut}(R[\mathbf{x}]/R)$ and $\sigma \in \bar{G} \cup J$ be such that $\deg_{\mathbf{w}} \phi \circ \sigma \leq \deg_{\mathbf{w}} \phi$. Then, the following statements hold:*
(i) *If $\deg_{\mathbf{w}} \phi(x_1) = \deg_{\mathbf{w}} \phi(x_2)$, then σ belongs to $\bar{G}$.*
(ii) *If $\deg_{\mathbf{w}} \phi(x_i) < \deg_{\mathbf{w}} \phi(x_j)$ for $i, j \in \{1, 2\}$ with $i \neq j$, then σ belongs to $J_{i,j}$.*

Proof. (i) In view of (1.5), we see that $J(R; x_i, x_j) \cap \mathrm{Aff}(R, \mathbf{x})$ is contained in $\mathrm{E}_0(R, \mathbf{x})$ for each $i, j \in \{1, 2\}$ with $i \neq j$. Since ι is affine, we have

$$J(R; x_i, x_j) \circ \iota \cap \mathrm{Aff}(R, \mathbf{x}) = \big(J(R; x_i, x_j) \cap \mathrm{Aff}(R, \mathbf{x})\big) \circ \iota.$$

Since ι belongs to $\mathrm{E}_0(R, \mathbf{x})$, the right-hand side of this equality is also contained in $\mathrm{E}_0(R, \mathbf{x})$. Hence, $J_{i,j} \cap \mathrm{Aff}(R, \mathbf{x})$ is contained in $\mathrm{E}_0(R, \mathbf{x})$. Thus, $J \cap \mathrm{Aff}(R, \mathbf{x})$ is contained in $\mathrm{E}_0(R, \mathbf{x})$, and therefore contained in $\bar{G}$.

Now, suppose to the contrary that σ does not belong to $\bar{G}$. Then, σ belongs to J. Since $J \cap \mathrm{Aff}(R, \mathbf{x})$ is contained in $\bar{G}$, it follows that σ does not belong to $\mathrm{Aff}(R, \mathbf{x})$. Write $\sigma(x_1)$ and $\sigma(x_2)$ as in (2.2), where $i, j \in \{1, 2\}$

with $i \neq j$. Then, we have $\deg_{x_i} h \geq 2$. Since $w_1 > 0$ or $w_2 > 0$, and $\delta := \deg_{\mathbf{w}} \phi(x_1) = \deg_{\mathbf{w}} \phi(x_2)$ by assumption, we get $\delta > 0$ by (1.6). Thus, we see from (2.3) that

$$\deg_{\mathbf{w}} \phi \circ \sigma = \deg_{\mathbf{w}} \phi(x_i) + \deg_{\mathbf{w}} \phi(h) \geq 3\delta > 2\delta = \deg_{\mathbf{w}} \phi,$$

a contradiction. Therefore, σ belongs to $\bar{G}$.

(ii) Since $\deg_{\mathbf{w}} \phi(x_i) < \deg_{\mathbf{w}} \phi(x_j) =: \delta$ by assumption, we have $\deg_{\mathbf{w}} \phi < 2\delta$, and $\delta > 0$ by (1.6). Suppose to the contrary that σ does not belong to $J_{i,j}$. Then, $\sigma(x_1)$ and $\sigma(x_2)$ do not belong to $R[x_i]$, and σ belongs to $\bar{G}$ or $J_{j,i}$. First, assume that σ belongs to $\bar{G}$. Then, σ is affine. Hence, $(\phi\circ\sigma)(x_l)$ is a linear polynomial in $\phi(x_i)$ and $\phi(x_j)$ over R for $l = 1, 2$. Moreover, $(\phi\circ\sigma)(x_l)$ does not belong to $R[\phi(x_i)]$, since $\sigma(x_l)$ does not belong to $R[x_i]$. Thus, we know that $\deg_{\mathbf{w}}(\phi \circ \sigma)(x_l) = \deg_{\mathbf{w}} \phi(x_j) = \delta$. Therefore, we get $\deg_{\mathbf{w}} \phi \circ \sigma = 2\delta > \deg_{\mathbf{w}} \phi$, a contradiction. Next, assume that σ belongs to $J_{j,i}$. Then, we may write $\sigma(x_p) = \alpha x_j - g$ and $\sigma(x_q) = \beta x_i - h$, where $p, q \in \{1, 2\}$ with $p \neq q$, $\alpha, \beta \in R^{\times}$, $g \in R$ and $h \in R[x_j]$. Since $\sigma(x_q)$ does not belong to $R[x_i]$, we have $\deg_{x_j} h \geq 1$. Hence, we know that $\deg_{\mathbf{w}} \phi \circ \sigma = \deg_{\mathbf{w}} \phi(x_j) + \deg_{\mathbf{w}} \phi(h) \geq 2\delta > \deg_{\mathbf{w}} \phi$, a contradiction. Therefore, σ belongs to $J_{i,j}$. □

Let W be the set of $\mathbf{w} \in \Gamma^2$ such that $w_1 > 0$ or $w_2 > 0$, and

$$\Sigma_{\mathbf{w}} := \{\deg_{\mathbf{w}} \psi \mid \psi \in \mathrm{Aut}(R[\mathbf{x}]/R)\}$$

is a well-ordered subset of Γ. If $w_i \geq 0$ for $i = 1, 2$, then $\{l_1 w_1 + l_2 w_2 \mid l_1, l_2 \in \mathbf{Z}_{\geq 0}\}$ is a well-ordered subset of Γ (cf. [**16**, Lemma 6.1]). Hence, $\Sigma_{\mathbf{w}}$ is also well-ordered. Thus, W contains the set of $\mathbf{w} \in \Gamma^2$ such that $w_1 > 0$ and $w_2 \geq 0$, or $w_1 \geq 0$ and $w_2 > 0$. Consider the following statements:

(I) Proposition 2.3 holds for each $\mathbf{w} \in W$.

(II) If $w_1 > 0$ or $w_2 > 0$ for $\mathbf{w} \in \Gamma^2$, then $\Sigma_{\mathbf{w}}$ is a well-ordered subset of Γ.

By the discussion after Proposition 2.3, (I) implies that Theorem 2.1 holds for each $\mathbf{w} \in W$. On the other hand, (II) implies that $\mathbf{w} \in \Gamma^2$ belongs to W if $w_1 > 0$ or $w_2 > 0$. Thus, Theorem 2.1 follows from (I) and (II). We prove (I) in the rest of this section. By making use of it, we prove (II) at the end of Section 1. To prove (I), we may assume that $\deg_{\mathbf{w}} \phi\circ\tau \leq \deg_{\mathbf{w}} \phi$ by the remark after Proposition 2.3. To prove (I) and (II), we may assume that $w_1 \leq w_2$ and $w_2 > 0$ by interchanging x_1 and x_2 if necessary.

Let us prove (I). Take any $\mathbf{w} \in W$. Then, $\Sigma_{\mathbf{w}}$ is a well-ordered subset of Γ. Since $\deg_{\mathbf{w}} \phi$ belongs to $\Sigma_{\mathbf{w}}$ for each $\phi \in G_{\mathbf{w}}$, we prove the statement of Proposition 2.3 by induction on $\deg_{\mathbf{w}} \phi$. By Lemma 1.1, $\mu := \min\{\deg_{\mathbf{w}} \phi \mid \phi \in G_{\mathbf{w}}\}$ is at least $|\mathbf{w}|$. Since $\mathrm{id}_{R[\mathbf{x}]}$ belongs to $G_{\mathbf{w}}$, we get $\mu = |\mathbf{w}|$. So assume that $\deg_{\mathbf{w}} \phi = |\mathbf{w}|$. Then, we have $\deg_{\mathbf{w}} \phi \circ \tau \leq |\mathbf{w}|$ by the assumption that $\deg_{\mathbf{w}} \phi \circ \tau \leq \deg_{\mathbf{w}} \phi$. This implies that $\deg_{\mathbf{w}} \phi \circ \tau = |\mathbf{w}|$ because of Lemma 1.1. Note that σ belongs to $G_{\mathbf{w}}$ if and only if σ belongs to $\bar{G} \cup J$ for $\sigma \in \mathrm{Aut}(R[\mathbf{x}]/R)$ with $\deg_{\mathbf{w}} \sigma = |\mathbf{w}|$. Hence, it suffices to show that $\phi\circ\tau$ belongs to $\bar{G} \cup J$. Since ϕ is an element of $G_{\mathbf{w}}$ with $\deg_{\mathbf{w}} \phi = |\mathbf{w}|$, we know that ϕ belongs to $\bar{G} \cup J$. More precisely, ϕ satisfies the following conditions.

Claim 2.5. *The following statements hold*:

(1) *If* $w_1 = w_2$, *then* $\deg_{\mathbf{w}} \phi(x_1) = \deg_{\mathbf{w}} \phi(x_2)$ *and* ϕ *belongs to* $\bar{G}$.

(2) *If* $w_1 < w_2$, *then one of the following conditions holds*:
(a) $\deg_{\mathbf{w}} \phi(x_1) < \deg_{\mathbf{w}} \phi(x_2)$ *and* ϕ *belongs to* $J(R; x_1, x_2)$.
(b) $\deg_{\mathbf{w}} \phi(x_2) < \deg_{\mathbf{w}} \phi(x_1)$ *and* ϕ *belongs to* $J(R; x_1, x_2) \circ \iota$.

PROOF. Take $(i, j) \in \{(1, 2), (2, 1)\}$ such that $\phi(x_j)$ does not belong to $R[x_1]$. Then, we have $\deg_{\mathbf{w}} \phi(x_j) \geq w_2$ by Lemma 1.2 (ii), since $w_2 > 0$ by assumption. Whether or not $\phi(x_i)$ belongs to $R[x_1]$, we have $\deg_{\mathbf{w}} \phi(x_i) \geq w_1$ by (i) and (ii) of Lemma 1.2, since $w_1 \leq w_2$ by assumption. Thus, we conclude that $\deg_{\mathbf{w}} \phi(x_i) = w_1$ and $\deg_{\mathbf{w}} \phi(x_j) = w_2$ from the assumption that $\deg_{\mathbf{w}} \phi = |\mathbf{w}|$.

(1) Since $w_1 = w_2$ by assumption, the first part follows from the preceding discussion. Since ϕ belongs to $\bar{G} \cup J$, we show that ϕ belongs to $\bar{G}$ when ϕ belongs to J. As remarked in the proof of Lemma 2.4, $J \cap \mathrm{Aff}(R, \mathbf{x})$ is contained in $\bar{G}$. So we show that ϕ belongs to $\mathrm{Aff}(R, \mathbf{x})$. By the definition of the $\mathbf{w}$-degree, we have $\deg_{\mathbf{w}} h = w_1 \deg h$ for each $h \in R[\mathbf{x}]$, since $w_1 = w_2$ and $w_2 > 0$. Hence, we get $w_1 \deg \phi(x_l) = \deg_{\mathbf{w}} \phi(x_l)$ for $l = 1, 2$. Since $\deg_{\mathbf{w}} \phi(x_l) = w_1$, it follows that $\deg \phi(x_l) = 1$ for $l = 1, 2$. Thus, ϕ belongs to $\mathrm{Aff}(R, \mathbf{x})$. Therefore, ϕ belongs to $\bar{G}$.

(2) We claim that $\phi(x_i)$ belongs to $R[x_1]$. In fact, if not, $w_1 = \deg_{\mathbf{w}} \phi(x_i)$ is not less than w_2 by Lemma 1.2 (ii), since $w_2 > 0$ by assumption. This contradicts that $w_1 < w_2$. If $(i, j) = (1, 2)$, then it follows that ϕ belongs to $J(R; x_1, x_2)$. Since $w_1 < w_2$, we know that $\deg_{\mathbf{w}} \phi(x_1) = \deg_{\mathbf{w}} \phi(x_i) = w_1$ is less than $\deg_{\mathbf{w}} \phi(x_2) = \deg_{\mathbf{w}} \phi(x_j) = w_2$. Therefore, ϕ satisfies (a). If $(i, j) = (2, 1)$, then $\phi(x_2) = \phi(x_i)$ belongs to $R[x_1]$. Hence, ϕ belongs to $J(R; x_1, x_2) \circ \iota$. Since $w_1 < w_2$, we know that $\deg_{\mathbf{w}} \phi(x_2) = \deg_{\mathbf{w}} \phi(x_i) = w_1$ is less than $\deg_{\mathbf{w}} \phi(x_1) = \deg_{\mathbf{w}} \phi(x_j) = w_2$. Therefore, ϕ satisfies (b). □

Now, we prove that $\phi \circ \tau$ belongs to $\bar{G} \cup J$. Since $\deg_{\mathbf{w}} \phi \circ \tau \leq \deg_{\mathbf{w}} \phi$, the statements (i) and (ii) of Lemma 2.4 hold for $\sigma = \tau$. If $w_1 = w_2$, then we know by Claim 2.5 (1) and Lemma 2.4 (i) that ϕ and τ belong to $\bar{G}$. Hence, $\phi \circ \tau$ belongs to $\bar{G}$. If $w_1 < w_2$, then we have (a) or (b) of Claim 2.5 (2). In the case of (a), τ belongs to $J_{1,2}$ by Lemma 2.4 (ii). Since ϕ belongs to $J(R; x_1, x_2)$, it follows that $\phi \circ \tau$ belongs to $J_{1,2}$. In the case of (b), τ belongs to $J_{2,1} = \iota \circ J_{1,2}$ by Lemma 2.4 (ii). Since ϕ belongs to $J(R; x_1, x_2) \circ \iota$, it follows that $\phi \circ \tau$ belongs to $J_{1,2}$. Hence, $\phi \circ \tau$ belongs to $\bar{G} \cup J$ in every case. Thus, $\phi \circ \tau$ belongs to $G_{\mathbf{w}}$. Therefore, the statement of Proposition 2.3 holds when $\deg_{\mathbf{w}} \phi = |\mathbf{w}|$.

Next, assume that $\deg_{\mathbf{w}} \phi > |\mathbf{w}|$. Since ϕ is an element of $G_{\mathbf{w}}$, we may find $\psi \in \bar{G} \cup J$ such that $\deg_{\mathbf{w}} \phi \circ \psi < \deg_{\mathbf{w}} \phi$ and $\phi \circ \psi$ belongs to $G_{\mathbf{w}}$ in view of (B). Then, $(\phi \circ \psi) \circ \sigma$ belongs to $G_{\mathbf{w}}$ for any $\sigma \in \bar{G} \cup J$ by induction assumption. We show that $\psi^{-1} \circ \tau$ belongs to $\bar{G} \cup J$. Then, it follows that $\phi \circ \tau = (\phi \circ \psi) \circ (\psi^{-1} \circ \tau)$ belongs to $G_{\mathbf{w}}$. If $\deg_{\mathbf{w}} \phi(x_1) = \deg_{\mathbf{w}} \phi(x_2)$, then ψ belongs to $\bar{G}$ by Lemma 2.4 (i) since $\deg_{\mathbf{w}} \phi \circ \psi < \deg_{\mathbf{w}} \phi$ by the choice of ψ. Since $\deg_{\mathbf{w}} \phi \circ \tau \leq \deg_{\mathbf{w}} \phi$ by assumption, τ also belongs to $\bar{G}$ by Lemma 2.4 (i). Hence, $\psi^{-1} \circ \tau$ belongs to $\bar{G}$. Likewise, if $\deg_{\mathbf{w}} \phi(x_i) < \deg_{\mathbf{w}} \phi(x_j)$ for some $i, j \in \{1, 2\}$ with $i \neq j$, then τ and ψ belong to $J_{i,j}$ by Lemma 2.4 (ii). Hence, $\psi^{-1} \circ \tau$ belongs to J by Lemma 2.2 (i). Therefore, $\psi^{-1} \circ \tau$ belongs to $\bar{G} \cup J$ in either case. This completes the proof of (I), and thereby proving Theorem 2.1 for each $\mathbf{w} \in W$. In particular, Theorem 2.1 is verified for each $\mathbf{w} \in \Gamma^2$ such that $w_1 > 0$ and $w_2 \geq 0$, or $w_1 \geq 0$ and $w_2 > 0$.

3. Analysis of reductions

Throughout this section, we assume that $n = 2$. We investigate properties of affine reductions and elementary reductions.

To deal with affine reductions, the following notion is crucial. Let K be the field of fractions of R. We define $V(R)$ to be the set of $\alpha/\beta \in K^\times$ for $\alpha, \beta \in R \setminus \{0\}$ such that $\alpha R + \beta R = R$. Note that $\alpha R + \beta R = R$ if and only if α and β are the entries of a row or column vector of an element of $GL(2, R)$ for $\alpha, \beta \in R$.

Lemma 3.1. (i) *$R \setminus \{0\}$ is contained in $V(R)$.*
(ii) *For $\alpha, \beta \in R \setminus \{0\}$, it holds that α/β belongs to $V(R)$ if and only if $\alpha R + \beta R$ is a principal ideal of R.*
(iii) *For $\gamma \in V(R)$ and $(a_{i,j})_{i,j} \in GL(2, R)$, it holds that*

$$\frac{a_{1,1}\gamma + a_{1,2}}{a_{2,1}\gamma + a_{2,2}}$$

belongs to $V(R)$.

Proof. (i) For each $a \in R \setminus \{0\}$, we have $aR + 1R = R$. Hence, $a = a/1$ belongs to $V(R)$.

(ii) Assume that α/β belongs to $V(R)$ for $\alpha, \beta \in R \setminus \{0\}$. Then, there exist $\alpha', \beta' \in R \setminus \{0\}$ such that $\alpha'/\beta' = \alpha/\beta$ and $\alpha' R + \beta' R = R$. Set $\gamma = \alpha/\alpha' = \beta/\beta'$ and take $a, b \in R$ such that $\alpha' a + \beta' b = 1$. Then, we have $\alpha a + \beta b = \gamma$. Hence, γ belongs to R, and γR is contained in $\alpha R + \beta R$. Since $\alpha = \gamma\alpha'$ and $\beta = \gamma\beta'$ belong to γR, we know that $\alpha R + \beta R$ is contained in γR. Thus, we get $\alpha R + \beta R = \gamma R$. Therefore, $\alpha R + \beta R$ is a principal ideal of R. Next, assume that $\alpha R + \beta R = \gamma R$ for some $\gamma \in R$. Then, α and β belong to γR. Hence, we have $\alpha = \alpha'\gamma$ and $\beta = \beta'\gamma$ for some $\alpha', \beta' \in R$. Then, we get $\alpha' R + \beta' R = R$. Therefore, $\alpha/\beta = \alpha'/\beta'$ belongs to $V(R)$.

(iii) Let $\gamma_1, \gamma_2 \in R \setminus \{0\}$ be such that $\gamma = \gamma_1/\gamma_2$ and $\gamma_1 R + \gamma_2 R = R$. Then, we have

$$\delta := \frac{a_{1,1}\gamma + a_{1,2}}{a_{2,1}\gamma + a_{2,2}} = \frac{a_{1,1}\gamma_1 + a_{1,2}\gamma_2}{a_{2,1}\gamma_1 + a_{2,2}\gamma_2}$$

and

$$(a_{1,1}\gamma_1 + a_{1,2}\gamma_2)R + (a_{2,1}\gamma_1 + a_{2,2}\gamma_2)R = \gamma_1 R + \gamma_2 R = R.$$

Hence, δ belongs to $V(R)$. □

By Lemma 3.1 (ii), it follows that $V(R) = K^\times$ if R is a PID. If r belongs to $V(R)$, then r^{-1} and ur belong to $V(R)$ for each $u \in R^\times$ by Lemma 3.1 (iii).

The following lemma naturally follows from the definition of affine reduction and elementary reduction.

Lemma 3.2. *Let $\phi, \tau \in \mathrm{Aut}(R[\mathbf{x}]/R)$ and $\mathbf{w} \in \Gamma^2$ be such that $\deg_{\mathbf{w}} \phi(x_i) > 0$ for $i = 1, 2$ and $\deg_{\mathbf{w}} \phi \circ \tau < \deg_{\mathbf{w}} \phi$.*
(i) *If τ belongs to $\mathrm{Aff}(R, \mathbf{x})$, then we have $\phi(x_1)^{\mathbf{w}} = a\phi(x_2)^{\mathbf{w}}$ for some $a \in V(R)$, and $(\phi \circ \tau)(x_i)^{\mathbf{w}} = b\phi(x_1)^{\mathbf{w}}$ for some $i \in \{1, 2\}$ and $b \in K^\times$.*
(ii) *Assume that τ belongs to $J_{i,j}$ for some $i, j \in \{1, 2\}$ with $i \neq j$. Then, we have $\phi(x_j)^{\mathbf{w}} = b(\phi(x_i)^{\mathbf{w}})^t$ for some $b \in R \setminus \{0\}$ and $t \geq 1$, and $(\phi \circ \tau)(x_p)^{\mathbf{w}} = \alpha\phi(x_i)^{\mathbf{w}}$ for some $p \in \{1, 2\}$ and $\alpha \in R^\times$.*

PROOF. (i) Take $A = (a_{i,j})_{i,j} \in GL(2, R)$ and $b_1, b_2 \in R$ such that

$$\tau(x_i) = a_{i,1}x_1 + a_{i,2}x_2 + b_i$$

for $i = 1, 2$, and put

$$\delta_i = \max\{\deg_{\mathbf{w}} a_{i,1}\phi(x_1), \deg_{\mathbf{w}} a_{i,2}\phi(x_2)\}$$

for $i = 1, 2$. Then, we have $\delta_i > 0$ for $i = 1, 2$, since no row of A is zero, and $\deg_{\mathbf{w}} \phi(x_j) > 0$ for $j = 1, 2$ by assumption. Since no column of A is zero, we know that $\delta_1 + \delta_2 \geq \deg_{\mathbf{w}} \phi$. We claim that $a_{j,1}\phi(x_1)^{\mathbf{w}} + a_{j,2}\phi(x_2)^{\mathbf{w}} = 0$ for some $j \in \{1, 2\}$. In fact, if not, we have

$$\deg_{\mathbf{w}}\big((\phi \circ \tau)(x_i) - b_i\big) = \deg_{\mathbf{w}}(a_{i,1}\phi(x_1) + a_{i,2}\phi(x_2)) = \delta_i$$

for $i = 1, 2$. Since b_i is an element of R, and δ_i is positive, this implies that $\deg_{\mathbf{w}}(\phi\circ\tau)(x_i) = \delta_i$ for $i = 1, 2$. Hence, we get $\deg_{\mathbf{w}} \phi\circ\tau = \delta_1+\delta_2 \geq \deg_{\mathbf{w}} \phi$, a contradiction. Thus, we may find $j \in \{1, 2\}$ as claimed. Then, $a_{j,1}$ and $a_{j,2}$ are nonzero, since $(a_{j,1}, a_{j,2}) \neq (0, 0)$, and $\phi(x_l)^{\mathbf{w}} \neq 0$ for $l = 1, 2$. Put $a = -a_{j,2}/a_{j,1}$. Then, a belongs to $V(R)$, and satisfies $\phi(x_1)^{\mathbf{w}} = a\phi(x_2)^{\mathbf{w}}$. Take $i \in \{1, 2\}$ with $i \neq j$. Then, $b := a_{i,1} + a_{i,2}a^{-1}$ belongs to $K^{\times}$, since $\det A \neq 0$. Hence, we get

$$a_{i,1}\phi(x_1)^{\mathbf{w}} + a_{i,2}\phi(x_2)^{\mathbf{w}} = a_{i,1}\phi(x_1)^{\mathbf{w}} + a_{i,2}(a^{-1}\phi(x_1)^{\mathbf{w}}) = b\phi(x_1)^{\mathbf{w}} \neq 0.$$

Since $\deg_{\mathbf{w}} \phi(x_1) = \deg_{\mathbf{w}} \phi(x_2)$, this implies that

$$\big((\phi \circ \tau)(x_i) - b_i\big)^{\mathbf{w}} = \big(a_{i,1}\phi(x_1) + a_{i,2}\phi(x_2)\big)^{\mathbf{w}} = b\phi(x_1)^{\mathbf{w}}.$$

Because b_i is a constant and $\deg_{\mathbf{w}} \phi(x_1) > 0$, it follows that $(\phi \circ \tau)(x_i)^{\mathbf{w}} = b\phi(x_1)^{\mathbf{w}}$.

(ii) Since τ is an element of $J_{i,j}$, we have an expression as (2.2) with $\sigma = \tau$. Then, we may write $(\phi \circ \tau)(x_p) = \alpha\phi(x_i) - g$. Since $\deg_{\mathbf{w}} \phi(x_i) > 0$ and g is a constant, it follows that $(\phi \circ \tau)(x_p)^{\mathbf{w}} = \alpha\phi(x_i)^{\mathbf{w}}$, proving the latter part. Consequently, we get $\deg_{\mathbf{w}}(\phi \circ \tau)(x_p) = \deg_{\mathbf{w}} \phi(x_i)$. Since $\deg_{\mathbf{w}} \phi \circ \tau < \deg_{\mathbf{w}} \phi$ by assumption, we know that the $\mathbf{w}$-degree of $(\phi \circ \tau)(x_q) = \beta\phi(x_j) - \phi(h)$ is less than $\deg_{\mathbf{w}} \phi(x_j)$. Because $\beta \neq 0$, this implies that $\beta\phi(x_j)^{\mathbf{w}} = \phi(h)^{\mathbf{w}}$. Since $\phi(h)$ belongs to $R[\phi(x_i)]$, it follows that $\phi(x_j)^{\mathbf{w}} = \beta^{-1}\phi(h)^{\mathbf{w}}$ belongs to $R[\phi(x_i)]^{\mathbf{w}} = R[\phi(x_i)^{\mathbf{w}}]$. Hence, we may write $\phi(x_j)^{\mathbf{w}} = b(\phi(x_i)^{\mathbf{w}})^t$ in view of the $\mathbf{w}$-homogeneity of $\phi(x_j)^{\mathbf{w}}$, where $b \in R \setminus \{0\}$ and $t \geq 0$. Then, we have $t \geq 1$, since $\deg_{\mathbf{w}} \phi(x_l) > 0$ for $l = 1, 2$. $\square$

From (i) and (ii) of Lemma 3.2, we get the following statement. Assume that $\phi \in \mathrm{Aut}(R[\mathbf{x}]/R)$, $\tau \in \mathrm{Aff}(R, \mathbf{x}) \cup J$ and $\mathbf{w} \in \Gamma^2$ satisfy $\deg_{\mathbf{w}} \phi(x_l) > 0$ for $l = 1, 2$ and $\deg_{\mathbf{w}} \phi \circ \tau < \deg_{\mathbf{w}} \phi$. Then, we have

$$\phi(x_j)^{\mathbf{w}} = a(\phi(x_i)^{\mathbf{w}})^t \quad \text{and} \quad \phi(x_i)^{\mathbf{w}} = b(\phi \circ \tau)(x_q)^{\mathbf{w}}$$

for some $i, j, q \in \{1, 2\}$ with $i \neq j$, $a, b \in K^{\times}$ and $t \geq 1$. Hence, $\phi(x_1)^{\mathbf{w}}$ and $\phi(x_2)^{\mathbf{w}}$ are powers of $(\phi \circ \tau)(x_q)^{\mathbf{w}}$ multiplied by elements of $K^{\times}$ for some $q \in \{1, 2\}$.

Now, let κ be any field, and $\mathbf{w} \in \Gamma^2$ such that $w_i > 0$ for $i = 1, 2$. Then, $\mathbf{w}$ belongs to the set W defined after Lemma 2.4. Hence, Proposition 2.3 holds for this $\mathbf{w}$ by (I). As a consequence, we know that $\mathrm{T}(\kappa, \mathbf{x}) = \mathrm{Aff}(\kappa, \mathbf{x})^{+}$ is contained in $\mathrm{Aff}(\kappa, \mathbf{x})_{\mathbf{w}}$. On the other hand, $\mathrm{T}(\kappa, \mathbf{x})$ is equal

to $\mathrm{Aut}(\kappa[\mathbf{x}]/\kappa)$ due to Jung [**12**] and van der Kulk [**14**]. Thus, $\mathrm{Aut}(\kappa[\mathbf{x}]/\kappa)$ is contained in $\mathrm{Aff}(\kappa, \mathbf{x})_{\mathbf{w}}$.

Lemma 3.3. *Assume that $w_i > 0$ for $i = 1, 2$. Then, for each $\phi \in \mathrm{Aut}(\kappa[\mathbf{x}]/\kappa)$ and $p \in \{1, 2\}$, there exist $a \in \kappa^{\times}$, $\psi \in \mathrm{Aff}(\kappa, \mathbf{x}) \cup J$ and $t \geq 1$ such that $\phi(x_p)^{\mathbf{w}} = a(\psi(x_1)^{\mathbf{w}})^t$.*

Proof. By the discussion above, ϕ belongs to $\mathrm{Aff}(\kappa, \mathbf{x})_{\mathbf{w}}$. Hence, there exist $l \in \mathbf{N}$, $\phi_1, \ldots, \phi_{l-1} \in \mathrm{Aut}(\kappa[\mathbf{x}]/\kappa)$ and $\phi_l \in T := \mathrm{Aff}(\kappa, \mathbf{x}) \cup J$ satisfying (A) and (B). We prove the lemma by induction on l. If $l = 1$, then we have $\phi = \phi_l$ by (A). Hence, ϕ belongs to T. Thus, if $p = 1$, then the statement holds for $a = 1$, $t = 1$ and $\psi = \phi$. Note that $(\phi \circ \iota)(x_1) = \phi(x_2)$ and $\phi \circ \iota$ belongs to T. Hence, if $p = 2$, then the statement holds for $a = 1$, $t = 1$ and $\psi = \phi \circ \iota$. Assume that $l \geq 2$. Then, for $q = 1, 2$, there exists $\psi_q \in T$ such that $\phi_2(x_q)^{\mathbf{w}}$ is a power of $\psi_q(x_1)^{\mathbf{w}}$ multiplied by a nonzero constant by induction assumption. By (B), there exists $\tau_1 \in T$ such that $\phi_2 = \phi \circ \tau_1$ and $\deg \phi \circ \tau_1 = \deg_{\mathbf{w}} \phi_2 < \deg_{\mathbf{w}} \phi$. Since $\phi(x_l)$ is not a constant, we have $\deg_{\mathbf{w}} \phi(x_l) > 0$ for $l = 1, 2$ by the choice of $\mathbf{w}$. Hence, we know from the remark after Lemma 3.2 that $\phi(x_1)^{\mathbf{w}}$ and $\phi(x_2)^{\mathbf{w}}$ are powers of $(\phi \circ \tau_1)(x_q)^{\mathbf{w}}$ multiplied by nonzero constants for some $q \in \{1, 2\}$. Therefore, $\phi(x_p)^{\mathbf{w}}$ is a power of $\psi_q(x_1)$ multiplied by a nonzero constant, since so is $(\phi \circ \tau_1)(x_q)^{\mathbf{w}} = \phi_2(x_q)^{\mathbf{w}}$. $\square$

CHAPTER 2

Tamely reduced coordinates

1. Structure of coordinates

Throughout this chapter, we assume that $n = 2$. The purpose of this chapter is to discuss reductions of polynomials by tame automorphisms. The notion of "tamely reduced" coordinates introduced in Section 2 will play a crucial role in the next two chapters.

In this section, we study the structure of coordinates. Let κ be any field. For each $f \in \kappa[\mathbf{x}] \setminus \{0\}$, we define an element $\mathbf{w}(f)$ of $(\mathbf{Z}_{\geq 0})^2$ by

$$\mathbf{w}(f) = (\deg_{x_2} f, \deg_{x_1} f).$$

Set $p_i = \deg_{x_i} f$ for $i = 1, 2$. Then, the monomial $x_1^{p_1} x_2^t$ appears in f for some $t \geq 0$. Hence, we have

$$\deg_{\mathbf{w}(f)} f \geq \deg_{\mathbf{w}(f)} x_1^{p_1} x_2^t = p_1 p_2 + t p_1 \geq p_1 p_2. \tag{1.1}$$

From this, we see that $x_1^{p_1}$ appears in f if $\deg_{\mathbf{w}(f)} f = p_1 p_2$ and $p_1 > 0$. If this is the case, then $x_1^{p_1}$ appears in $f^{\mathbf{w}(f)}$, since $\deg_{\mathbf{w}(f)} x_1^{p_1} = p_1 p_2 = \deg_{\mathbf{w}(f)} f$. Likewise, if $\deg_{\mathbf{w}(f)} f = p_1 p_2$ and $p_2 > 0$, then $x_2^{p_2}$ appears in $f^{\mathbf{w}(f)}$.

Lemma 1.1. *Assume that $p_i > 0$ and x_j^q appears in $f^{\mathbf{w}(f)}$ for some $i, j \in \{1, 2\}$ with $i \neq j$ and $q \geq 0$. Then, we have $q = p_j$ and $\deg_{\mathbf{w}(f)} f = p_1 p_2$. Hence, both $x_1^{p_1}$ and $x_2^{p_2}$ appear in $f^{\mathbf{w}(f)}$.*

PROOF. Since x_j^q appears in $f^{\mathbf{w}(f)}$, we have $\deg_{\mathbf{w}(f)} f = \deg_{\mathbf{w}(f)} x_j^q = p_i q$. Hence, we get $p_i q \geq p_i p_j$ in view of (1.1). Since $p_i > 0$ by assumption, it follows that $q \geq p_j$. On the other hand, we have $q \leq \deg_{x_j} f = p_j$, since x_j^q is a monomial appearing in f. Thus, we get $q = p_j$, and therefore $\deg_{\mathbf{w}(f)} f = p_i q = p_1 p_2$. Since $p_i > 0$, this implies that $x_i^{p_i}$ appears in $f^{\mathbf{w}(f)}$ by the remark. Consequently, both $x_1^{p_1}$ and $x_2^{p_2}$ appear in $f^{\mathbf{w}(f)}$. $\square$

Now, assume that f is a coordinate of $\kappa[\mathbf{x}]$ over κ. Then, we have

$$|\mathbf{w}(f)| = \deg_{x_2} f + \deg_{x_1} f \geq 1,$$

since f is not a constant. We remark that $|\mathbf{w}(f)| = 1$ if and only if f is a linear polynomial in x_i over κ for some $i \in \{1, 2\}$, and hence if and only if f belongs to $\kappa[x_i]$ for some $i \in \{1, 2\}$.

Proposition 1.2. *Let f be a coordinate of $\kappa[\mathbf{x}]$ over κ with $|\mathbf{w}(f)| > 1$. Then, there exist $i, j \in \{1, 2\}$ with $i \neq j$, $l, m \in \mathbf{N}$ and $a, b \in \kappa^\times$ such that*

$$\deg_{x_i} f = m, \quad \deg_{x_j} f = lm \quad \text{and} \quad f^{\mathbf{w}(f)} = a(x_i - b x_j^l)^m.$$

PROOF. Since $|\mathbf{w}(f)| > 1$ by assumption, f does not belong to $\kappa[x_i]$ for $i = 1, 2$. Hence, we have $p_i := \deg_{x_i} f > 0$ for $i = 1, 2$. By Lemma 3.3, we know that $f^{\mathbf{w}(f)} = c(\psi(x_1)^{\mathbf{w}(f)})^m$ for some $c \in \kappa^\times$, $\psi \in \mathrm{Aff}(\kappa, \mathbf{x}) \cup J$ and $m \geq 1$. Then, $\psi(x_1)^{\mathbf{w}(f)}$ has the form αx_i^l or $\alpha(x_i + bx_j^l)$ for some $\alpha, b \in \kappa^\times$, $l \geq 1$ and $i, j \in \{1, 2\}$ with $i \neq j$. Thus, $f^{\mathbf{w}(f)}$ is written as ax_i^{lm} or $a(x_i + bx_j^l)^m$, where $a := c\alpha^m$. If $f^{\mathbf{w}(f)} = ax_i^{lm}$, then the two monomials $x_1^{p_1}$ and $x_2^{p_2}$ appear in $f^{\mathbf{w}(f)}$ by Lemma 1.1, a contradiction. Therefore, we conclude that $f^{\mathbf{w}(f)} = a(x_i + bx_j^l)^m$. Since x_i^m and x_j^{lm} appear in $f^{\mathbf{w}(f)}$, it follows from Lemma 1.1 that $\deg_{x_i} f = p_i = m$ and $\deg_{x_j} f = p_j = lm$. $\square$

Next, let $f \in \kappa[\mathbf{x}] \setminus \kappa$ be such that $p_i := \deg_{x_i} f > 0$ for $i = 1, 2$. Then, the following conditions are equivalent:

(a) $\deg_{\mathbf{w}(f)} f = p_1 p_2$.

(b) $x_1^{p_1}$ and $x_2^{p_2}$ appear in $f^{\mathbf{w}(f)}$.

In fact, since $p_i > 0$ for $i = 1, 2$ by assumption, (a) implies (b) by the remark after (1.1), and the converse is obvious. In view of Proposition 1.2, we see that the equivalent conditions are satisfied if f is a coordinate of $\kappa[\mathbf{x}]$ over κ with $|\mathbf{w}(f)| > 1$.

Put
$$q_i = \frac{p_i}{\gcd(p_1, p_2)}$$
for $i = 1, 2$. Then, every $\mathbf{w}(f)$-homogeneous element of $\kappa[\mathbf{x}]$ is written as a product of a monomial and irreducible binomials of the form $x_1^{q_1} - bx_2^{q_2}$ for some $b \in \bar{\kappa}^\times$. Here, $\bar{\kappa}$ denotes an algebraic closure of κ. Hence, if f satisfies (a) and (b), then we may write

$$f^{\mathbf{w}(f)} = a \prod_{i=1}^{m} (x_1^{q_1} - b_i x_2^{q_2}) \tag{1.2}$$

because of (b) and the $\mathbf{w}(f)$-homogeneity of $f^{\mathbf{w}(f)}$, where $a \in \kappa^\times$, $m \in \mathbf{N}$ and $b_i \in \bar{\kappa}^\times$ for $i = 1, \ldots, m$. Then, we have $p_i = mq_i$ for $i = 1, 2$.

In this situation, we have the following proposition.

Proposition 1.3. *Let f be as above, and let $\mathbf{w} \in \Gamma^2$ and $\tau \in \mathrm{Aut}(\kappa[\mathbf{x}]/\kappa)$ be such that $w_1 > 0$ or $w_2 > 0$, and $(\tau(x_1)^{\mathbf{w}})^{q_1} \neq b_i(\tau(x_2)^{\mathbf{w}})^{q_2}$ for $i = 1, \ldots, m$. Then, we have*

$$\deg_{\mathbf{w}} \tau(f) = \max\{p_1 \deg_{\mathbf{w}} \tau(x_1), p_2 \deg_{\mathbf{w}} \tau(x_2)\}. \tag{1.3}$$

PROOF. We may extend τ to an element of $\mathrm{Aut}(\bar{\kappa}[\mathbf{x}]/\bar{\kappa})$ naturally. Then, we have
$$\tau(f^{\mathbf{w}(f)}) = a \prod_{i=1}^{m} (\tau(x_1)^{q_1} - b_i \tau(x_2)^{q_2}).$$
Put $\xi_i = \deg_{\mathbf{w}} \tau(x_i)$ for $i = 1, 2$. Then, we obtain
$$\deg_{\mathbf{w}} (\tau(x_1)^{q_1} - b_i \tau(x_2)^{q_2}) = \max\{q_1 \xi_1, q_2 \xi_2\}$$
for $i = 1, \ldots, m$ by the assumption that $(\tau(x_1)^{\mathbf{w}})^{q_1} \neq b_i(\tau(x_2)^{\mathbf{w}})^{q_2}$ and $b_i \neq 0$. Hence, it follows that
$$\deg_{\mathbf{w}} \tau(f^{\mathbf{w}(f)}) = m \max\{q_1 \xi_1, q_2 \xi_2\} = \max\{p_1 \xi_1, p_2 \xi_2\} =: d.$$

Thus, it suffices to check that $\deg_{\mathbf{w}} \tau(f - f^{\mathbf{w}(f)}) < d$. Let $x_1^{a_1} x_2^{a_2}$ be any monomial appearing in $f - f^{\mathbf{w}(f)}$. Then, we have

$$a_1 p_2 + a_2 p_1 = \deg_{\mathbf{w}(f)} x_1^{a_1} x_2^{a_2} < \deg_{\mathbf{w}(f)} f = p_1 p_2,$$

since f satisfies the condition (a) by assumption. This yields that

$$a_2 < p_2 - a_1 p_2 p_1^{-1} \quad \text{and} \quad a_1 < p_1 - a_2 p_1 p_2^{-1}. \tag{1.4}$$

First, assume that $p_1 \xi_1 \leq p_2 \xi_2$. Then, we have $d = p_2 \xi_2$. Since $w_1 > 0$ or $w_2 > 0$ by assumption, we know by (1.6) that $\max\{\xi_1, \xi_2\} > 0$. Hence, we get $\xi_2 > 0$. Thus, it follows from the first inequality of (1.4) that

$$\begin{aligned} \deg_{\mathbf{w}} \tau(x_1^{a_1} x_2^{a_2}) &= a_1 \xi_1 + a_2 \xi_2 < a_1 \xi_1 + (p_2 - a_1 p_2 p_1^{-1}) \xi_2 \\ &= a_1 p_1^{-1} (p_1 \xi_1 - p_2 \xi_2) + p_2 \xi_2 \leq p_2 \xi_2 = d. \end{aligned}$$

Next, assume that $p_1 \xi_1 > p_2 \xi_2$. Then, we have $d = p_1 \xi_1$, and $\xi_1 > 0$ by (1.6) as above. Hence, it follows from the second inequality of (1.4) that

$$\begin{aligned} \deg_{\mathbf{w}} \tau(x_1^{a_1} x_2^{a_2}) &= a_1 \xi_1 + a_2 \xi_2 < (p_1 - a_2 p_1 p_2^{-1}) \xi_1 + a_2 \xi_2 \\ &= a_2 p_2^{-1} (p_2 \xi_2 - p_1 \xi_1) + p_1 \xi_1 < p_1 \xi_1 = d. \end{aligned}$$

Thus, we have proved $\deg_{\mathbf{w}} \tau(f - f^{\mathbf{w}(f)}) < d$. Therefore, we get $\deg_{\mathbf{w}} \tau(f) = d$. □

Let f be a coordinate of $\kappa[\mathbf{x}]$ over κ with $|\mathbf{w}(f)| > 1$. Then, f satisfies the assumption of Proposition 1.3. Write $f^{\mathbf{w}(f)}$ as in Proposition 1.2, and take $\mathbf{w} \in \Gamma^2$ such that $w_1 > 0$ or $w_2 > 0$. Clearly, $x_i^{\mathbf{w}}$ is not equal to $b(x_j^{\mathbf{w}})^l$. Hence, we get

$$\deg_{\mathbf{w}} f = \max\{m \deg_{\mathbf{w}} x_i, lm \deg_{\mathbf{w}} x_j\} = \max\{m w_i, lm w_j\} \tag{1.5}$$

by applying Proposition 1.3 with $\tau = \mathrm{id}_{\kappa[\mathbf{x}]}$.

Now, we prove the statement (II) after Lemma 2.4. We may assume that $w_1 < 0$ and $w_2 > 0$ as noted. First, we show that $\deg_{\mathbf{w}} f$ belongs to $\{t w_2 \mid t \in \mathbf{N}\}$ for each coordinate f of $R[\mathbf{x}]$ over R not belonging to $R[x_1]$. Since $w_2 > 0$ by assumption, this is clear if f belongs to $R[x_2]$. Assume that f does not belong to $R[x_2]$. Then, we have $|\mathbf{w}(f)| > 1$, since f also does not belong to $R[x_1]$ by assumption. Note that f is regarded as a coordinate of $K[\mathbf{x}]$ over K. Hence, we know by (1.5) that $\deg_{\mathbf{w}} f = \max\{t_1 w_1, t_2 w_2\}$ for some $t_1, t_2 \in \mathbf{N}$. Since $w_1 < 0$ and $w_2 > 0$, we get $\deg_{\mathbf{w}} f = t_2 w_2$. Thus, $\deg_{\mathbf{w}} f$ belongs to $\{t w_2 \mid t \in \mathbf{N}\}$. Now, observe that

$$\{t w_2 \mid t \in \mathbf{N}\} \cup \{w_1 + w_2\} \tag{1.6}$$

is a well-ordered subset of Γ. We show that $\Sigma_{\mathbf{w}}$ is contained in (1.6). Take any $\phi \in \mathrm{Aut}(R[\mathbf{x}]/R)$. If $\phi(x_i)$ does not belong to $R[x_1]$ for $i = 1, 2$, then $\deg_{\mathbf{w}} \phi(x_i)$ belongs to $\{t w_2 \mid t \in \mathbf{N}\}$ for $i = 1, 2$ by the preceding discussion. Hence, $\deg_{\mathbf{w}} \phi$ belongs to (1.6). Assume that $\phi(x_i)$ belongs to $R[x_1]$ for some $i \in \{1, 2\}$. Then, $\deg_{\mathbf{w}} \phi(x_i)$ is equal to zero or w_1 by Lemma 1.2 (i). Because ϕ belongs to $J_{1,2}$, we have $\phi(x_j) = a x_2 + g$ for some $a \in R^{\times}$ and $g \in R[x_1]$ for $j \neq i$. Hence, we get $\deg_{\mathbf{w}} \phi(x_j) = w_2$, since $\deg_{\mathbf{w}} a x_2 = w_2 > 0$ and $\deg_{\mathbf{w}} g = (\deg_{x_1} g) w_1 \leq 0$. Thus, $\deg_{\mathbf{w}} \phi$ is equal to w_2 or $w_1 + w_2$, and

so belongs to (1.6). Therefore, $\Sigma_{\mathbf{w}}$ is contained in (1.6), proving the well-orderedness of $\Sigma_{\mathbf{w}}$. This completes the proof of (II), and thereby completing the proof of Theorem 2.1.

2. Reduction of a polynomial

Let f be an element of $R[\mathbf{x}] \setminus R$. Recall that f is said to be *tamely reduced* over R if

$$|\mathbf{w}(\tau(f))| \geq |\mathbf{w}(f)|$$

holds for every $\tau \in \mathrm{T}(R, \mathbf{x})$. Obviously, f is tamely reduced over R if $|\mathbf{w}(f)| = 1$, since f is a linear polynomial in x_i over R for some $i \in \{1, 2\}$.

Now, assume that $p_i := \deg_{x_i} f > 0$ for $i = 1, 2$, and f satisfies the equivalent conditions (a) and (b) before Proposition 1.3. Then, $f^{\mathbf{w}(f)}$ is written as in (1.2) with $\kappa = K$.

With this notation, we have the following proposition.

Proposition 2.1. *Let $f \in R[\mathbf{x}]$ be as above. Then, the following conditions are equivalent*:
(1) *f is tamely reduced over R.*
(2) *The following statements hold*:
(i) *If $p_1 = p_2$, then $b_1, \ldots, b_m$ do not belong to $V(R)$.*
(ii) *If $p_1 < p_2$ and p_1 divides p_2, then $b_1, \ldots, b_m$ do not belong to R.*
(iii) *If $p_1 > p_2$ and p_2 divides p_1, then $b_1^{-1}, \ldots, b_m^{-1}$ do not belong to R.*
(3) *Let Γ be any totally ordered additive group, $\mathbf{w} \in \Gamma^2$ such that $w_1 > 0$ or $w_2 > 0$, and τ any element of $\mathrm{T}(R, \mathbf{x})$. Then, we have*

$$\deg_{\mathbf{w}} \tau(f) = \max\{p_i \deg_{\mathbf{w}} \tau(x_i) \mid i = 1, 2\}.$$

To prove this proposition, we use the following lemmas.

Lemma 2.2. *Let f be as above, and $g \in R[\mathbf{x}]$ such that $\deg_{\mathbf{w}(f)} g \leq \deg_{\mathbf{w}(f)} f$ and $\deg_{x_i} g^{\mathbf{w}(f)} < \deg_{x_i} f$ for some $i \in \{1, 2\}$. Then, we have $|\mathbf{w}(g)| < |\mathbf{w}(f)|$.*

Proof. It suffices to show that $\deg_{x_i} g < p_i$ and $\deg_{x_j} g \leq p_j$, where $j \neq i$. Suppose to the contrary that $\deg_{x_i} g \geq p_i$. Then, the monomial $x_i^{p_i+s} x_j^t$ appears in g for some $s, t \geq 0$. Since $p_l > 0$ for $l = 1, 2$ by assumption, we have

$$p_i p_j \leq (p_i + s)p_j + tp_i = \deg_{\mathbf{w}(f)} x_i^{p_i+s} x_j^t \leq \deg_{\mathbf{w}(f)} g \leq \deg_{\mathbf{w}(f)} f.$$

On the other hand, we know that $\deg_{\mathbf{w}(f)} f = p_i p_j$ by the condition (a) before Proposition 1.3. Hence, it follows that $s = t = 0$ and $\deg_{\mathbf{w}(f)} x_i^{p_i} = \deg_{\mathbf{w}(f)} g$. Thus, $x_i^{p_i}$ appears in $g^{\mathbf{w}(f)}$. Therefore, we get $\deg_{x_i} g^{\mathbf{w}(f)} \geq p_i = \deg_{x_i} f$, a contradiction. This proves that $\deg_{x_i} g < p_i$. Next, suppose to the contrary that $\deg_{x_j} g > p_j$. Then, the monomial $x_i^s x_j^{p_j+t}$ appears in g for some $s \geq 0$ and $t \geq 1$. Hence, we have

$$\deg_{\mathbf{w}(f)} g \geq \deg_{\mathbf{w}(f)} x_i^s x_j^{p_j+t} = sp_j + (p_j + t)p_i > p_i p_j = \deg_{\mathbf{w}(f)} f,$$

a contradiction. Therefore, we get $\deg_{x_j} g \leq p_j$. $\square$

We say that $\tau \in \mathrm{Aut}(R[\mathbf{x}]/R)$ is $\mathbf{w}$-*homogeneous* if $\tau(x_i)$ is $\mathbf{w}$-homogeneous and $\deg_{\mathbf{w}} \tau(x_i) = w_i$ for each i. If this is the case, then $\tau(R[\mathbf{x}]_\gamma)$ is contained in $R[\mathbf{x}]_\gamma$ for each $\gamma \in \Gamma$. Hence, we have $\deg_{\mathbf{w}} \tau(h) = \deg_{\mathbf{w}} h$ and $\tau(h)^{\mathbf{w}} = \tau(h^{\mathbf{w}})$ for each $h \in R[\mathbf{x}]$.

Lemma 2.3. *Let f be as above, and assume that $\tau \in \mathrm{Aut}(R[\mathbf{x}]/R)$ is $\mathbf{w}(f)$-homogeneous. If $\deg_{x_i} \tau(f^{\mathbf{w}(f)}) < \deg_{x_i} f^{\mathbf{w}(f)}$ for some $i \in \{1, 2\}$, then we have $|\mathbf{w}(\tau(f))| < |\mathbf{w}(f)|$.*

PROOF. Since τ is $\mathbf{w}(f)$-homogeneous, we have $\deg_{\mathbf{w}(f)} \tau(f) = \deg_{\mathbf{w}(f)} f$ and $\tau(f)^{\mathbf{w}(f)} = \tau(f^{\mathbf{w}(f)})$. Since $\deg_{x_i} \tau(f^{\mathbf{w}(f)}) < \deg_{x_i} f^{\mathbf{w}(f)}$ by assumption, it follows that $\deg_{x_i} \tau(f)^{\mathbf{w}(f)} < \deg_{x_i} f^{\mathbf{w}(f)} \leq \deg_{x_i} f$. Therefore, we get $|\mathbf{w}(\tau(f))| < |\mathbf{w}(f)|$ by applying Lemma 2.2 with $g = \tau(f)$. $\square$

Now, let us prove Proposition 2.1. First, we prove that (1) implies the three statements of (2) by contradiction. Suppose that $p_1 = p_2$ and b_s belongs to $V(R)$ for some s. Then, we have $q_1 = q_2 = 1$. Since b_s belongs to $V(R)$, there exists $(a_{i,j})_{i,j} \in GL(2, R)$ such that $b_s = a_{1,1}/a_{2,1}$. Define $\tau \in \mathrm{Aff}(R, \mathbf{x})$ by $\tau(x_i) = a_{i,1}x_1 + a_{i,2}x_2$ for $i = 1, 2$. Then, τ is $\mathbf{w}(f)$-homogeneous, and satisfies $\deg_{x_1} \tau(x_1 - b_i x_2) \leq 1$ for $i = 1, \dots, m$. By the choice of $(a_{i,j})_{i,j}$, we have

$$\tau(x_1 - b_s x_2) = a_{1,1}x_1 + a_{1,2}x_2 - b_s(a_{2,1}x_1 + a_{2,2}x_2) = (a_{1,2} - b_s a_{2,2})x_2.$$

Hence, we know that

$$\deg_{x_1} \tau(f^{\mathbf{w}(f)}) = \deg_{x_1} a \prod_{i=1}^{m} \tau(x_1 - b_i x_2) < m = \deg_{x_1} f^{\mathbf{w}(f)}.$$

Thus, we conclude that $|\mathbf{w}(\tau(f))| < |\mathbf{w}(f)|$ by Lemma 2.3. Since τ is affine, this contradicts that f is tamely reduced over R. Therefore, (1) implies (i) of (2).

Next, suppose that $p_1 < p_2$, p_1 divides p_2, and b_s belongs to R for some s. Then, we have $q_1 = 1$ and $q_2 = p_2/p_1$. Since b_s is an element of R, we may define $\tau \in \mathrm{Aut}(R[\mathbf{x}]/R[x_2])$ by $\tau(x_1) = x_1 + b_s x_2^{q_2}$. Then, τ is $\mathbf{w}(f)$-homogeneous, and

$$\tau(f^{\mathbf{w}(f)}) = a \prod_{i=1}^{m} \tau(x_1 - b_i x_2^{q_2}) = a \prod_{i=1}^{m} \bigl(x_1 + (b_s - b_i)x_2^{q_2}\bigr).$$

From this, we know that $\deg_{x_2} \tau(f^{\mathbf{w}(f)}) < mq_2 = \deg_{x_2} f^{\mathbf{w}(f)}$. Thus, we conclude that $|\mathbf{w}(\tau(f))| < |\mathbf{w}(f)|$ by Lemma 2.3. Since τ is elementary, this contradicts that f is tamely reduced over R. Therefore, (1) implies (ii) of (2). We can check that (1) implies (iii) of (2) similarly.

Next, we prove that (2) implies (3). Take any totally ordered additive group Γ, $\mathbf{w} \in \Gamma^2$ with $w_1 > 0$ or $w_2 > 0$, and any $\tau \in \mathrm{T}(R, \mathbf{x})$. Then, we may regard τ as an element of $\mathrm{Aut}(K[\mathbf{x}]/K)$. Hence, we may use Proposition 1.3 for $\mathbf{w}$ and τ by taking $\kappa = K$, since $w_1 > 0$ or $w_2 > 0$ by assumption. Thus, it suffices to check that $(\tau(x_1)^{\mathbf{w}})^{q_1} \neq b_i(\tau(x_2)^{\mathbf{w}})^{q_2}$ for $i = 1, \dots, m$. Suppose to the contrary that

$$(\tau(x_1)^{\mathbf{w}})^{q_1} = b_s(\tau(x_2)^{\mathbf{w}})^{q_2} \tag{2.1}$$

for some s. Then, $\tau(x_1)^{\mathbf{w}}$ and $\tau(x_2)^{\mathbf{w}}$ are algebraically dependent over K. Hence, we have $\deg_{\mathbf{w}} \tau > |\mathbf{w}|$ by Lemma 1.1. Since τ belongs to $\mathrm{T}(R, \mathbf{x}) = \mathrm{Aff}(R, \mathbf{x})^+$, and $w_1 > 0$ or $w_2 > 0$ by assumption, it follows from Theorem 2.1 that τ admits an affine reduction or elementary reduction for the weight $\mathbf{w}$.

First, assume that τ admits an affine reduction. Since $w_1 > 0$ or $w_2 > 0$, we have $\deg_{\mathbf{w}} \tau(x_i) > 0$ for some $i \in \{1, 2\}$ by (1.6), and hence for any $i \in \{1, 2\}$ by (2.1). On account of Lemma 3.2 (i) with $\phi = \tau$, we know that $\tau(x_1)^{\mathbf{w}} = c\tau(x_2)^{\mathbf{w}}$ for some $c \in V(R)$. Then, we have $c^{q_1}(\tau(x_2)^{\mathbf{w}})^{q_1} = b_s(\tau(x_2)^{\mathbf{w}})^{q_2}$ in view of (2.1). Since $\deg_{\mathbf{w}} \tau(x_2) > 0$, it follows that $q_1 = q_2$. This implies that $q_1 = q_2 = 1$. Hence, we get $c = b_s$. Thus, b_s belongs to $V(R)$. On the other hand, we have $p_1 = p_1$ since $q_1 = q_2$. This contradicts (i) of (2).

Next, assume that τ admits an elementary reduction. Then, by Lemma 3.2 (ii) with $\phi = \tau$, we know that $\tau(x_i)^{\mathbf{w}} = c(\tau(x_j)^{\mathbf{w}})^t$ for some $i, j \in \{1, 2\}$ with $i \neq j$, $c \in R \setminus \{0\}$ and $t \geq 1$. Note that c and c^{-1} belong to $V(R)$. Hence, we can get a contradiction as in the previous case when $t = 1$. Assume that $t \geq 2$. If $(i, j) = (1, 2)$, then we have $c^{q_1}(\tau(x_2)^{\mathbf{w}})^{tq_1} = b_s(\tau(x_2)^{\mathbf{w}})^{q_2}$. Since $\deg_{\mathbf{w}} \tau(x_2) > 0$, this gives that $tq_1 = q_2$. Hence, we get $q_1 = 1$, $q_2 = t \geq 2$ and $b_s = c$. Thus, we know that $p_1 < p_2$, p_1 divides p_2, and b_s belongs to R. This contradicts (ii) of (2). If $(i, j) = (2, 1)$, then we have $(\tau(x_1)^{\mathbf{w}})^{q_1} = b_s c^{q_2}(\tau(x_1)^{\mathbf{w}})^{tq_2}$. Since $\deg_{\mathbf{w}} \tau(x_1) > 0$, this gives that $q_1 = tq_2$. Hence, we get $q_2 = 1$, $q_1 = t \geq 2$ and $b_s^{-1} = c$. Thus, we know that $p_1 > p_2$, p_2 divides p_1, and b_s^{-1} belongs to R. This contradicts (iii) of (2). Therefore, $(\tau(x_1)^{\mathbf{w}})^{q_1} \neq b_i(\tau(x_2)^{\mathbf{w}})^{q_2}$ holds for $i = 1, \ldots, m$. This proves that (2) implies (3).

Finally, we prove that (3) implies (1). Without loss of generality, we may assume that $p_1 \leq p_2$ by interchanging x_1 and x_2 if necessary. Take any $\tau \in \mathrm{T}(R, \mathbf{x})$. Then, we have

$$\deg_{x_j} \tau(f) = \max\{p_i \deg_{x_j} \tau(x_i) \mid i = 1, 2\} \tag{2.2}$$

for $j = 1, 2$ by applying (3) with $\Gamma = \mathbf{Z}$ and $\mathbf{w} = \mathbf{e}_j$. First, take $s \in \{1, 2\}$ such that $\deg_{x_s} \tau(x_2) \geq 1$. Then, we obtain $\deg_{x_s} \tau(f) \geq p_2$ from (2.2) with $j = s$. Next, take $l \in \{1, 2\}$ such that $\deg_{x_t} \tau(x_l) \geq 1$ for $t \in \{1, 2\}$ with $t \neq s$. Then, we get $\deg_{x_t} \tau(f) \geq p_l$ by (2.2) with $j = t$. Since $p_1 \leq p_2$ by assumption, it follows that $\deg_{x_t} \tau(f) \geq p_1$. Hence, we have

$$|\mathbf{w}(\tau(f))| = \deg_{x_s} \tau(f) + \deg_{x_t} \tau(f) \geq p_2 + p_1 = |\mathbf{w}(f)|.$$

Thus, f is tamely reduced over R. Therefore, (3) implies (1). This completes the proof of Proposition 2.1.

For $f \in R[\mathbf{x}] \setminus R$, we consider the subgroup

$$H(f) := \mathrm{Aut}(R[\mathbf{x}]/R[f]) \cap \mathrm{T}(R, \mathbf{x})$$

of $\mathrm{Aut}(R[\mathbf{x}]/R)$. It is worthwhile to mention that, if $R = k[x_3]$, then we have

$$H(f) = \mathrm{Aut}(k[x_1, x_2, x_3]/k[x_3, f]) \cap \mathrm{T}(k, \{x_1, x_2, x_3\}) \tag{2.3}$$

by virtue of Theorem 1. Here, we recall that k is an arbitrary field of characteristic zero throughout this monograph.

By means of Proposition 2.1, we obtain the following theorem.

Theorem 2.4. *Assume that* $f \in R[\mathbf{x}]$ *satisfies* $\deg_{x_i} f > 0$ *for* $i = 1, 2$ *and the equivalent conditions* (a) *and* (b) *before Proposition 1.3. If* f *is tamely reduced over* R, *then the following assertions hold*:
(i) *If* $\deg_{x_1} f = \deg_{x_2} f$, *then* $H(f)$ *is contained in* $\mathrm{Aff}(R, \mathbf{x})$.
(ii) *If* $\deg_{x_i} f > \deg_{x_j} f$ *for* $i, j \in \{1, 2\}$ *with* $i \neq j$, *then* $H(f)$ *is contained in* $J(R; x_i, x_j)$.

Proof. Since f satisfies the assumption of Proposition 2.1, and is tamely reduced over R, we know that f satisfies the equivalent conditions of Proposition 2.1. In particular, (2.2) holds for every $\tau \in \mathrm{T}(R, \mathbf{x})$ due to (3).

Now, take any $\tau \in H(f)$. Then, τ belongs to $\mathrm{T}(R, \mathbf{x})$ by definition. Hence, τ satisfies (2.2). Since $\tau(f) = f$, it follows that

$$\deg_{x_j} f = \max\{(\deg_{x_i} f) \deg_{x_j} \tau(x_i) \mid i = 1, 2\} \tag{2.4}$$

for $j = 1, 2$.

(i) Since $\deg_{x_1} f = \deg_{x_2} f$, we get $\max\{\deg_{x_j} \tau(x_i) \mid i = 1, 2\} = 1$ for $j = 1, 2$ by (2.4). Put $\mathbf{w}_i = \mathbf{w}(\tau(x_i))$ for $i = 1, 2$. Then, it follows that $|\mathbf{w}_i| = 1$ or $\mathbf{w}_i = (1, 1)$. If $|\mathbf{w}_i| = 1$, then $\tau(x_i)$ is a linear polynomial. The same holds when $\mathbf{w}_i = (1, 1)$ in view of Proposition 1.3. Thus, τ belongs to $\mathrm{Aff}(R, \mathbf{x})$. Therefore, $H(f)$ is contained in $\mathrm{Aff}(R, \mathbf{x})$.

(ii) Since $\deg_{x_i} f > \deg_{x_j} f$, we have $\deg_{x_i} f > 0$, and

$$\deg_{x_i} f > \deg_{x_j} f \geq (\deg_{x_i} f) \deg_{x_j} \tau(x_i)$$

by (2.4). Hence, we get $\deg_{x_j} \tau(x_i) < 1$, and so $\tau(x_i)$ belongs to $R[x_i]$. Thus, τ belongs to $J(R; x_i, x_j)$. Therefore, $H(f)$ is contained in $J(R; x_i, x_j)$. $\square$

Now, let S be an over domain of R. In the rest of this section, we consider the case where $f \in R[\mathbf{x}]$ is a coordinate of $S[\mathbf{x}]$ over S. Assume that $|\mathbf{w}(f)| > 1$. Then, f satisfies $\deg_{x_i} f > 0$ for $i = 1, 2$ and the equivalent conditions (a) and (b) before Proposition 1.3. Hence, f satisfies the assumption of Proposition 2.1. Let L be the field of fractions of S. Then, we may regard f as a coordinate of $L[\mathbf{x}]$ over L. Hence, there exist $i, j \in \{1, 2\}$ with $i \neq j$, $a, b \in L^\times$ and $l, m \in \mathbf{N}$ such that

$$\deg_{x_i} f = m, \quad \deg_{x_j} f = lm \quad \text{and} \quad f^{\mathbf{w}(f)} = a(x_i - b x_j^l)^m$$

by Proposition 1.2 with $\kappa = L$.

With this notation, the following proposition follows from Proposition 2.1.

Proposition 2.5. *Let* $f \in R[\mathbf{x}]$ *be a coordinate of* $S[\mathbf{x}]$ *over* S *such that* $|\mathbf{w}(f)| > 1$. *Then, the following assertions hold*:
(i) *Assume that* $\deg_{x_1} f = \deg_{x_2} f$, *i.e.,* $l = 1$. *Then,* f *is tamely reduced over* R *if and only if* b *does not belong to* $V(R)$.
(ii) *Assume that* $\deg_{x_i} f < \deg_{x_j} f$, *i.e.,* $l \geq 2$. *Then,* f *is tamely reduced over* R *if and only if* b *does not belong to* R.

In fact, we obtain (i) and (ii) from the equivalence between (1) and (2) of Proposition 2.1, since $\deg_{x_i} f = m$ divides $\deg_{x_j} f = lm$ in the case of (ii).

Corollary 2.6. *Assume that* $V(R) = L^{\times}$. *Let* $f \in R[\mathbf{x}]$ *be a coordinate of* $S[\mathbf{x}]$ *over* S. *If* f *is tamely reduced over* R, *then we have* $\deg_{x_1} f \neq \deg_{x_2} f$.

Proof. Suppose to the contrary that $\deg_{x_1} f = \deg_{x_2} f$. Then, we have $|\mathbf{w}(f)| > 1$, since f is not a constant. Hence, $f^{\mathbf{w}(f)}$ is written as above. Since b is an element of $L^{\times}$, and $L^{\times} = V(R)$ by assumption, it follows that b belongs to $V(R)$. Thus, f is not tamely reduce over R by Proposition 2.5 (i). This is a contradiction. Therefore, we get $\deg_{x_1} f \neq \deg_{x_2} f$. □

Theorem 2.4 immediately implies the following theorem. This theorem plays a crucial role in Chapters 3 and 4.

Theorem 2.7. *Let* $f \in R[\mathbf{x}]$ *be a coordinate of* $S[\mathbf{x}]$ *over* S *with* $|\mathbf{w}(f)| > 1$. *If* f *is tamely reduced over* R, *then the following assertions hold*:
(i) *If* $\deg_{x_1} f = \deg_{x_2} f$, *then* $H(f)$ *is contained in* $\mathrm{Aff}(R, \mathbf{x})$.
(ii) *If* $\deg_{x_i} f > \deg_{x_j} f$ *for* $i, j \in \{1, 2\}$ *with* $i \neq j$, *then* $H(f)$ *is contained in* $J(R; x_i, x_j)$.

CHAPTER 3

Triangularizability and tameness

1. Triangularizability

Throughout this chapter, we assume that R is a $\mathbf{Q}$-domain, and K is the field of fractions of R. We study tameness and wildness of $\exp D$ for $D \in \mathrm{LND}_R\, R[\mathbf{x}]$ using the theory developed in the previous chapter.

The following theorem is a key result in this section. This theorem gives a solution to Problem 5 for $n = 2$.

Theorem 1.1. *Assume that $n = 2$. Let $D \in \mathrm{LND}_R\, R[\mathbf{x}]$ be such that $\exp D$ belongs to $\mathrm{T}(R, \mathbf{x})$. Then, there exists $\tau \in \mathrm{T}(R, \mathbf{x})$ such that $\tau^{-1} \circ D \circ \tau$ is triangular or affine. If $V(R) = K^\times$, then there exists $\tau \in \mathrm{T}(R, \mathbf{x})$ such that $\tau^{-1} \circ D \circ \tau$ is triangular.*

Here, we recall that $D \in \mathrm{LND}_R\, R[\mathbf{x}]$ is said to be *affine* if $\deg D(x_i) \leq 1$ for $i = 1, \ldots, n$.

In view of the remark after Problem 5, Theorem 1.1 implies the following theorem. From this theorem, we know that the answer to Question 1 is affirmative when $n = 2$.

Theorem 1.2. *Assume that $n = 2$. Let D be an element of $\mathrm{LND}_R\, R[\mathbf{x}]$. If $\exp fD$ belongs to $\mathrm{T}(R, \mathbf{x})$ for some $f \in \ker D \setminus \{0\}$, then $\exp D$ belongs to $\mathrm{T}(R, \mathbf{x})$.*

With the aid of Theorem 1, we get the following theorem from Theorems 1.1 and 1.2. This theorem gives partial answers to Question 1 and Problem 5.

Theorem 1.3. *Assume that $n = 3$. Let D be an element of $\mathrm{LND}_k\, k[\mathbf{x}]$ which kills a tame coordinate of $k[\mathbf{x}]$ over k. Then, the following assertions hold:*
(i) *$\exp D$ belongs to $\mathrm{T}(k, \mathbf{x})$ if and only if $\tau^{-1} \circ D \circ \tau$ is triangular for some $\tau \in \mathrm{T}(k, \mathbf{x})$.*
(ii) *If $\exp fD$ belongs to $\mathrm{T}(k, \mathbf{x})$ for some $f \in \ker D \setminus \{0\}$, then $\exp D$ belongs to $\mathrm{T}(k, \mathbf{x})$.*

Proof. By assumption, there exists a tame coordinate g of $k[\mathbf{x}]$ over k such that $D(g) = 0$. Take $\sigma \in \mathrm{T}(k, \mathbf{x})$ such that $\sigma(x_1) = g$, and put $D' = \sigma^{-1} \circ D \circ \sigma$. Then, we have $D'(x_1) = 0$. Hence, D' belongs to $\mathrm{LND}_{k[x_1]}\, k[\mathbf{x}]$, and $\exp D'$ belongs to $\mathrm{Aut}(k[\mathbf{x}]/k[x_1])$.

(i) The "if" part is clear. We prove the "only if" part. Assume that $\exp D$ belongs to $\mathrm{T}(k, \mathbf{x})$. Then, $\exp D'$ belongs to $\mathrm{T}(k, \mathbf{x})$. Since $\exp D'$ belongs to $\mathrm{Aut}(k[\mathbf{x}]/k[x_1])$, it follows that $\exp D'$ belongs to $\mathrm{T}(k[x_1], \{x_2, x_3\})$ on account of Theorem 1. Since $k[x_1]$ is a PID, we have $V(k[x_1]) = k(x_1)^\times$.

Regard D' as a derivation of the polynomial ring in the two variables x_2 and x_3 over $k[x_1]$. Then, it follows from the last part of Theorem 1.1 that

$$D'' := \tau^{-1} \circ D' \circ \tau = (\sigma \circ \tau)^{-1} \circ D \circ (\sigma \circ \tau)$$

is triangular for some $\tau \in \mathrm{T}(k[x_1], \{x_2, x_3\})$, i.e., $D''(x_1) = 0$, $D''(x_2)$ belongs to $k[x_1]$, and $D''(x_3)$ belongs to $k[x_1, x_2]$. This implies that D'' is a triangular derivation of $k[\mathbf{x}]$ over k. Clearly, $\sigma \circ \tau$ belongs to $\mathrm{T}(k, \mathbf{x})$. Therefore, the "only if" part is true.

(ii) Since $\exp fD$ belongs to $\mathrm{T}(k, \mathbf{x})$ by assumption, and σ is an element of $\mathrm{T}(R, \mathbf{x})$, we see that

$$\exp \sigma^{-1}(f)D' = \sigma^{-1} \circ (\exp fD) \circ \sigma$$

belongs to $\mathrm{T}(k, \mathbf{x})$. We may regard $\sigma^{-1}(f)D'$ as an element of $\mathrm{LND}_{k[x_1]} k[\mathbf{x}]$. Hence, $\exp \sigma^{-1}(f)D'$ belongs to $\mathrm{Aut}(k[\mathbf{x}]/k[x_1])$. Thanks to Theorem 1, it follows that $\exp \sigma^{-1}(f)D'$ belongs to $\mathrm{T}(k[x_1], \{x_2, x_3\})$. Thus, we know that $\exp D'$ belongs to $\mathrm{T}(k[x_1], \{x_2, x_3\})$ by virtue of Theorem 1.2. This implies that $\exp D'$ belongs to $\mathrm{T}(k, \mathbf{x})$. Therefore, $\exp D$ belongs to $\mathrm{T}(k, \mathbf{x})$. □

Now, we prove Theorem 1.1. The following is a key proposition.

Proposition 1.4. *Assume that $n = 2$. Let $f \in R[\mathbf{x}]$ be a coordinate of $S[\mathbf{x}]$ over S for some over domain S of R, and $D \in \mathrm{LND}_R R[\mathbf{x}]$ such that $D(f) = 0$ and $\exp D$ belongs to $\mathrm{T}(R, \mathbf{x})$. If f is tamely reduced over R, then the following assertions hold:*
(i) *If $\deg_{x_1} f > \deg_{x_2} f$, then D is triangular.*
(ii) *If $\deg_{x_1} f = \deg_{x_2} f$, then D is affine.*

Let f and D be as in Proposition 1.4. Then, we have $(\exp D)(f) = f$, since $D(f) = 0$ by assumption. Hence, $\exp D$ belongs to $H(f)$ due to the assumption that $\exp D$ belongs to $\mathrm{T}(R, \mathbf{x})$. Since f is tamely reduced over R, we know by Theorem 2.7 that $H(f)$ is contained in $J(R; x_1, x_2)$ in the case of (i), and in $\mathrm{Aff}(R, \mathbf{x})$ in the case of (ii). Thus, $\exp D$ belongs to $J(R; x_1, x_2)$ in the case of (i), and to $\mathrm{Aff}(R, \mathbf{x})$ in the case of (ii). Therefore, Proposition 1.4 is proved by the following lemma.

Lemma 1.5. *For each $D \in \mathrm{LND}_R R[\mathbf{x}]$, the following assertions hold, where $n \in \mathbf{N}$ may be arbitrary.*
(i) *If $\exp D$ belongs to $J(R; x_1, \dots, x_n)$, then D is triangular.*
(ii) *If $\exp D$ belongs to $\mathrm{Aff}(R, \mathbf{x})$, then D is affine.*

Proof. Consider the power series

$$p(z) = \exp z - 1 = \sum_{i=1}^{\infty} \frac{z^i}{i!},$$

where z is a variable. Since $z = \log\big((\exp z - 1) + 1\big)$, we get the identity

$$z = \sum_{i=1}^{\infty} (-1)^{i+1} \frac{p(z)^i}{i} \tag{1.1}$$

in the formal power series ring $\mathbf{Q}[[z]]$, where the right-hand side of (1.1) makes sense because $p(z)$ is an element of $z\mathbf{Q}[[z]]$. Now, let $\mathrm{End}_R R[\mathbf{x}]$ be the ring of the R-linear endomorphisms of $R[\mathbf{x}]$, $\mathbf{Q}[D]$ the $\mathbf{Q}$-subalgebra of

$\mathrm{End}_R R[\mathbf{x}]$ generated by D, and $\mathbf{Q}[[D]]$ the completion of $\mathbf{Q}[D]$ by the maximal ideal of $\mathbf{Q}[D]$ generated by D. Then, $\mathbf{Q}[[D]]$ is contained in $\mathrm{End}_R R[\mathbf{x}]$ by the assumption that D is locally nilpotent. Since $p(D)$ is a multiple of D, we see that $p(D)$ is locally nilpotent, i.e., for each $f \in R[\mathbf{x}]$, there exists $l \in \mathbf{N}$ such that $p(D)^l(f) = 0$.

(i) Since $\exp D$ belongs to $J(R; x_1, \ldots, x_n)$ by assumption, we may write

$$(\exp D)(x_i) = a_i x_i + g_i$$

for $i = 1, \ldots, n$, where $a_i \in R^\times$ and $g_i \in R[x_1, \ldots, x_{i-1}]$. Then, it is easy to check that $(\exp D)^j(x_i)$ has the form $a_i^j x_i + g_{i,j}$ for some $g_{i,j} \in R[x_1, \ldots, x_{i-1}]$ for each $j \geq 0$ by induction on j. Hence, we get

$$\begin{aligned} p(D)^l(x_i) &= (\exp D - \mathrm{id}_{R[\mathbf{x}]})^l(x_i) \\ &= \sum_{j=0}^{l} (-1)^{l-j} \binom{l}{j} (\exp D)^j(x_i) = \sum_{j=0}^{l} (-1)^{l-j} \binom{l}{j} a_i^j x_i + h_{i,l} = (a_i - 1)^l x_i + h_{i,l} \end{aligned}$$

for each $l \in \mathbf{N}$, where $h_{i,l} \in R[x_1, \ldots, x_{i-1}]$. Since $p(D)$ is locally nilpotent, it follows that $a_i = 1$ for each i. Hence, $p(D)^l(x_i) = h_{i,l}$ belongs to $R[x_1, \ldots, x_{i-1}]$. In view of (1.1) with z replaced by D, we know that $D(x_i)$ belongs to $R[x_1, \ldots, x_{i-1}]$ for each i. Therefore, D is triangular.

(ii) Since $\exp D$ is affine by assumption, $p(D)(x_i) = (\exp D)(x_i) - x_i$ has total degree at most one for $i = 1, \ldots, n$. Hence, we have $\deg p(D)^l(x_i) \leq 1$ for each $l \in \mathbf{N}$. In view of (1.1) with z replaced by D, we get $\deg D(x_i) \leq 1$ for each i. Therefore, D is affine. $\square$

We mention that the method used in the proof of this lemma is similar to that used in van den Essen [**6**, Proposition 2.1.3] and Nowicki [**23**, Proposition 6.1.4 (6)].

Now, let us complete the proof of Theorem 1.1. If $D = 0$, then the theorem is clear. Assume that $D \neq 0$. Let $\tilde{D}$ be the natural extension of D to an element of $\mathrm{LND}_K K[\mathbf{x}]$. Since R is a $\mathbf{Q}$-domain, K is of characteristic zero. Hence, there exists a coordinate f of $K[\mathbf{x}]$ over K such that $\ker \tilde{D} = K[f]$ by Theorem 2. Multiplying by an element of $K^\times$, we may assume that f belongs to $R[\mathbf{x}]$. Take $\tau \in \mathrm{T}(R, \mathbf{x})$ such that $|\mathbf{w}(\tau(f))|$ is minimal. Then, $f' := \tau(f)$ remains a coordinate of $K[\mathbf{x}]$ over K, and is tamely reduced over R. Without loss of generality, we may assume that $\deg_{x_1} f' \geq \deg_{x_2} f'$ by changing τ if necessary. Put $D' := \tau \circ D \circ \tau^{-1}$. Then, D' kills f'. Therefore, by applying Proposition 1.4 with $S = K$, we know that D' is triangular if $\deg_{x_1} f' > \deg_{x_2} f'$, and affine if $\deg_{x_1} f' = \deg_{x_2} f'$. This proves the first part of Theorem 1.1.

To prove the last part, assume that $V(R) = K^\times$. Then, we have $\deg_{x_1} f' \neq \deg_{x_2} f'$ thanks to Lemma 2.6, because f' is tamely reduced over R. Since $\deg_{x_1} f' \geq \deg_{x_2} f'$ by assumption, it follows that $\deg_{x_1} f' > \deg_{x_2} f'$. Therefore, D' is triangular by Proposition 1.4 (i), proving the last part. This completes the proof of Theorem 1.1.

2. Nagata type wild automorphisms

In this section, we study Problem 2. First, assume that $n = 2$, and let D be a triangular derivation of $R[\mathbf{x}]$ over R. We consider when $\exp fD$

belongs to $\mathrm{T}(R,\mathbf{x})$ for $f \in \ker D$. If f belongs to R, then fD is triangular. Hence, $\exp fD$ belongs to $\mathrm{T}(R,\mathbf{x})$. If $D(x_i) = 0$ for some $i \in \{1,2\}$, then $\exp fD$ belongs to $\mathrm{Aut}(R[\mathbf{x}]/R[x_i])$, and hence belongs to $\mathrm{T}(R,\mathbf{x})$ for any $f \in \ker D$.

Assume that f does not belong to R, and $D(x_i) \neq 0$ for $i = 1,2$. Write

$$D(x_1) = a \quad \text{and} \quad D(x_2) = \sum_{i=0}^{l} b_i x_1^i, \tag{2.1}$$

where $l \in \mathbf{Z}_{\geq 0}$, and $a, b_0, \ldots, b_l \in R$ with $a \neq 0$ and $b_l \neq 0$. Then,

$$h := a x_2 - \sum_{i=0}^{l} \frac{b_i}{i+1} x_1^{i+1} \tag{2.2}$$

is a coordinate of $K[\mathbf{x}]$ over K such that $D(h) = 0$. By the following theorem, it follows that $\ker D$ is contained in $K[h]$.

Theorem 2.1 (Rentschler)**.** *Assume that $n = 2$. If $D(f) = 0$ holds for $D \in \mathrm{LND}_k\, k[\mathbf{x}] \setminus \{0\}$ and a coordinate f of $k[\mathbf{x}]$ over k, then we have* $\ker D = k[f]$.

We remark that this theorem is a consequence of Theorem 2. Indeed, we have $\ker D = k[g]$ for some coordinate g of $k[\mathbf{x}]$ over k by Theorem 2. Hence, if f is a coordinate of $k[\mathbf{x}]$ over k belonging to $\ker D$, then f is a linear polynomial in g over k. Therefore, we get $k[f] = k[g] = \ker D$.

Since f is an element of $\ker D$, it follows that f belongs to $K[h]$. We denote by $\deg_h f$ the degree of f as a polynomial in h over K. Then, it holds that $\deg_h f = \deg_{x_2} f$, since $\deg_{x_2} h = 1$.

We define

$$I = \{ i \in \{0, \ldots, l\} \mid b_i \notin aR \}. \tag{2.3}$$

Then, we have the following theorem. This theorem gives a complete solution to Problem 2 in the case of $n = 2$.

Theorem 2.2. *Let D be as above, and f an element of* $\ker D \setminus R$. *Then,* $\exp fD$ *belongs to* $\mathrm{T}(R,\mathbf{x})$ *if and only if one of the following conditions holds*:

(1) $I = \emptyset$. (2) $I = \{0\}$, *and b_0/a belongs to $V(R)$ or* $\deg_{x_2} f = 1$.

In particular, if $V(R) = K^\times$, then $\exp fD$ *belongs to* $\mathrm{T}(R,\mathbf{x})$ *if and only if b_i belongs to aR for $i = 1, \ldots, l$.*

Proof. Define $\tau \in \mathrm{Aut}(R[\mathbf{x}]/R[x_1])$ by

$$\tau(x_2) = x_2 + \sum_{i \in I'} \frac{b_i}{(i+1)a} x_1^{i+1}, \tag{2.4}$$

where $I' := \{0, \ldots, l\} \setminus I$. Then, we have

$$h_0 := \tau(h) = a x_2 - \sum_{i \in I} \frac{b_i}{(i+1)} x_1^{i+1}. \tag{2.5}$$

Put $D_0 = \tau \circ D \circ \tau^{-1}$ and $f_0 = \tau(f)$. Then, $\phi := \exp fD$ is tame if and only if $\phi_0 := \exp f_0 D_0 = \tau \circ \phi \circ \tau^{-1}$ is tame. Note that $D_0(h_0) = \tau(D(h)) = 0$

and $D_0(x_1) = \tau(D(\tau^{-1}(x_1))) = \tau(D(x_1)) = a$. Hence, we have $\phi_0(h_0) = h_0$ and

$$\phi_0(x_1) = (\exp f_0 D_0)(x_1) = x_1 + f_0 D_0(x_1) = x_1 + a f_0. \tag{2.6}$$

Now, assume that $I \neq \emptyset$ and $I \neq \{0\}$, i.e., $t := \max I \geq 1$. Then, we prove that ϕ is wild. Suppose to the contrary that ϕ is tame. Then, ϕ_0 is tame. Since $\phi_0(h_0) = h_0$, it follows that ϕ_0 belongs to $H(h_0)$. From (2.5), we see that $\deg_{x_1} h_0 = t+1 > 1 = \deg_{x_2} h_0$, and

$$h_0^{\mathbf{w}(h_0)} = a(x_2 - b x_1^{t+1}), \quad \text{where} \quad b := \frac{b_t}{(t+1)a}. \tag{2.7}$$

Since t is an element of I, we know that b_t does not belong to aR, and so b does not belong to R. Hence, h_0 is tamely reduced over R by Proposition 2.5 (ii). By Theorem 2.7 (ii), it follows that $H(h_0)$ is contained in $J(R; x_1, x_2)$. Thus, ϕ_0 belongs to $J(R; x_1, x_2)$. Because of (2.6), this implies that $a f_0$ belongs to $R[x_1]$. Since $D_0(f_0) = 0$ and $D_0(x_1) = a \neq 0$, we conclude that f_0 belongs to R. Thus, f belongs to R, a contradiction. Therefore, ϕ is wild if $I \neq \emptyset$ and $I \neq \{0\}$.

Next, assume that $I = \emptyset$. Then, we have $h_0 = a x_2$. Since $\phi_0(h_0) = h_0$, it follows that ϕ_0 belongs to $\mathrm{Aut}(R[\mathbf{x}]/R[x_2])$. Hence, ϕ_0 is elementary. Therefore, ϕ is tame.

Finally, assume that $I = \{0\}$. Then, we have $h_0 = a x_2 - b_0 x_1$ with $b_0 \neq 0$. Hence, the total degree of f_0 is equal to $\deg_{h_0} f_0 = \deg_h f = \deg_{x_2} f$. Since $h_0 = \phi_0(h_0)$, we have

$$a x_2 - b_0 x_1 = h_0 = \phi_0(h_0) = \phi_0(a x_2 - b_0 x_1) = a\phi_0(x_2) - b_0(x_1 + a f_0)$$

in view of (2.6). This gives that $\phi_0(x_2) = x_2 + b_0 f_0$. We prove that ϕ is tame if and only if $\deg_{x_2} f = 1$ or b_0/a belongs to $V(R)$. First, assume that $\deg_{x_2} f = 1$. Then, we have $\deg f_0 = 1$ as mentioned. Hence, we get $\deg \phi_0(x_i) = 1$ for $i = 1, 2$. Thus, ϕ_0 is affine. Therefore, ϕ is tame. Next, assume that b_0/a belongs to $V(R)$. Then, there exists $\sigma \in \mathrm{Aff}(R, \mathbf{x})$ such that $\sigma(h_0) = c x_2$ for some $c \in R \setminus \{0\}$. Put $\phi_1 = \sigma \circ \phi_0 \circ \sigma^{-1}$. Then, we have $\phi_1(c x_2) = \sigma(\phi_0(h_0)) = \sigma(h_0) = c x_2$. Hence, ϕ_1 belongs to $\mathrm{Aut}(R[\mathbf{x}]/R[x_2])$. Thus, ϕ_1 is elementary. Therefore, ϕ is tame. Finally, assume that $\deg_{x_2} f \neq 1$ and b_0/a does not belong to $V(R)$. Then, we have $\deg_{x_2} f \geq 2$, since f is not an element of R by assumption. Hence, we get $\deg f_0 \geq 2$, and so ϕ_0 is not affine in view of (2.6). Since b_0/a does not belong to $V(R)$, we know by Proposition 2.5 (i) that $h_0 = a x_2 - b_0 x_1$ is tamely reduced over R. Hence, $H(h_0)$ is contained in $\mathrm{Aff}(R, \mathbf{x})$ by Theorem 2.7 (i). Thus, ϕ_0 does not belong to $H(h_0)$. Since $\phi_0(h_0) = h_0$, this implies that ϕ_0 is wild. Therefore, ϕ is wild. This proves that ϕ is tame if and only if (1) or (2) holds.

To prove the last part, assume that $V(R) = K^\times$. Then, we claim that (2) is equivalent to $I = \{0\}$. In fact, if $I = \{0\}$, then we have $b_0 \neq 0$, and hence b_0/a always belongs to $K^\times = V(R)$. Therefore, the first part of the theorem implies that ϕ is tame if and only if $I = \emptyset$ or $I = \{0\}$. By the definition of I, this condition is equivalent to the condition that b_i belongs to aR for $i = 1, \ldots, l$. $\square$

Next, assume that $n = 3$, and let D be a triangular derivation of $k[\mathbf{x}]$ over k. We consider when $\exp fD$ belongs to $\mathrm{T}(k,\mathbf{x})$ for $f \in \ker D \setminus k$.

(i) Assume that $D(x_1) = 0$ and f belongs to $k[x_1]$. Then, fD is triangular. Hence, $\exp fD$ belongs to $\mathrm{T}(k,\mathbf{x})$.

(ii) Assume that $D(x_i) = D(x_j) = 0$ for some $1 \leq i < j \leq 3$. Then, $\exp fD$ belongs to $\mathrm{Aut}(k[\mathbf{x}]/k[x_i,x_j])$. Therefore, $\exp fD$ belongs to $\mathrm{T}(k,\mathbf{x})$.

(iii) Assume that $D(x_1) \neq 0$. Then, $D(x_1)$ belongs to $k^\times$ by the triangularity of D. Hence, $s := x_1/D(x_1)$ is an element of $k[x_1]$. Define $\tau \in J(k[x_1]; x_2, x_3)$ by

$$\tau(x_i) = \sum_{l \geq 0} \frac{D^l(x_i)}{l!}(-s)^l$$

for $i = 2, 3$. Then, we have $D(\tau(x_i)) = 0$ for $i = 2, 3$. In fact, since $D(s) = 1$, it follows that

$$D\left(\frac{D^l(x_i)}{l!}(-s)^l\right) = g_{l+1} - g_l \text{ for each } l \geq 0, \text{ where } g_l := l\frac{D^l(x_i)}{l!}(-s)^{l-1}.$$

Hence, we know that $D_0 := \tau^{-1} \circ fD \circ \tau$ kills x_i for $i = 2, 3$. Thus, $\exp D_0$ belongs to $\mathrm{Aut}(k[\mathbf{x}]/k[x_2,x_3])$. Therefore, $\exp fD$ belongs to $\mathrm{T}(k,\mathbf{x})$.

In the rest of the case, tameness of $\exp fD$ is determined by the following theorem.

Theorem 2.3. *Assume that $n = 3$. Let D be a triangular derivation of $k[\mathbf{x}]$ over k such that $D(x_1) = 0$ and $D(x_i) \neq 0$ for $i = 2, 3$, and f an element of $\ker D \setminus k[x_1]$. Then, $\exp fD$ belongs to $\mathrm{T}(k,\mathbf{x})$ if and only if $\partial D(x_3)/\partial x_2$ belongs to $D(x_2)k[x_1,x_2]$.*

By the triangularity of D, we may regard $D(x_3)$ as a polynomial in x_2 over $k[x_1]$. Then, $\partial D(x_3)/\partial x_2$ belongs to $D(x_2)k[x_1,x_2]$ if and only if the coefficient of x_2^i in $D(x_3)$ belongs to $D(x_2)k[x_1]$ for each $i \geq 1$.

Theorem 2.3 is derived from Theorem 2.2 with the aid of Theorem 1 as follows. Let $R = k[x_1]$, $y_i = x_{i+1}$ for $i = 1, 2$ and $\mathbf{y} = \{y_1, y_2\}$. Then, we may regard D as a triangular derivation of $R[\mathbf{y}]$ over R. By assumption, f does not belong to $R = k[x_1]$, and $D(y_i) = D(x_{i+1}) \neq 0$ for $i = 1, 2$. Hence, D fulfills the assumption of Theorem 2.2. Since $k[x_1]$ is a PID, we have $V(k[x_1]) = k(x_1)^\times$. Thus, we know by the last part of Theorem 2.2 that $\phi := \exp fD$ belongs to $\mathrm{T}(R,\mathbf{y})$ if and only if the coefficient of $y_1^i = x_2^i$ in $D(y_2) = D(x_3)$ belongs to $D(y_1)R = D(x_2)k[x_1]$ for each $i \geq 1$. This condition is equivalent to the condition that $\partial D(x_3)/\partial x_2$ belongs to $D(x_2)k[x_1,x_2]$ as remarked. Thanks to Theorem 1, ϕ belongs to $\mathrm{T}(R,\mathbf{y}) = \mathrm{T}(k[x_1],\{x_2,x_3\})$ if and only if ϕ belongs to $\mathrm{T}(k,\mathbf{x})$, since ϕ is an element of $\mathrm{Aut}(K[\mathbf{x}]/k[x_1])$. Therefore, ϕ belongs to $\mathrm{T}(k,\mathbf{x})$ if and only if $\partial D(x_3)/\partial x_2$ belongs to $D(x_2)k[x_1,x_2]$. This proves Theorem 2.3.

As an application of Theorem 2.3, we describe all the wild automorphisms of $k[\mathbf{x}]$ over k of the form $\exp fD$ for some triangular derivation D of $k[\mathbf{x}]$ over k and $f \in \ker D$. Let Λ be the set of $(g,h) \in (k[x_1] \setminus \{0\}) \times (x_2k[x_1,x_2] \setminus \{0\})$ as follows:

(i) g and h have no common factor;

(ii) g is a monic polynomial in x_1;

(iii) $\partial^2 h/\partial x_2^2$ does not belong to $gk[x_1, x_2]$.

Here, by "no common factor", we mean "no non-constant common factor". For each $(g, h) \in \Lambda$, we define a triangular derivation $T_{g,h}$ of $k[\mathbf{x}]$ over k by

$$T_{g,h}(x_1) = 0, \quad T_{g,h}(x_2) = g, \quad T_{g,h}(x_3) = -\frac{\partial h}{\partial x_2}. \tag{2.8}$$

Then, we have

$$T_{g,h}(gx_3 + h) = gT_{g,h}(x_3) + \frac{\partial h}{\partial x_2} T_{g,h}(x_2) = 0.$$

Hence, $k[x_1, gx_3 + h]$ is contained in $\ker T_{g,h}$. Take any $f \in k[x_1, gx_3 + h] \setminus k[x_1]$. Then, it follows from Theorem 2.3 that

$$\Phi^f_{g,h} := \exp fT_{g,h}$$

does not belong to $\mathrm{T}(k, \mathbf{x})$, since $\partial T_{g,h}(x_3)/\partial x_2 = -\partial^2 h/\partial x_2^2$ does not belong to $T_{g,h}(x_2)k[x_1, x_2] = gk[x_1, x_2]$ by (iii).

Proposition 2.4. *Assume that $n = 3$. Let D be a triangular derivation of $k[\mathbf{x}]$ over k and $f \in \ker D$ such that $\exp fD$ does not belong to $\mathrm{T}(k, \mathbf{x})$. Then, there exist unique $(g, h) \in \Lambda$ and $f_0 \in k[x_1, gx_3 + h] \setminus k[x_1]$ such that $fD = f_0 T_{g,h}$.*

Proof. First, we prove the existence of g, h and f_0. Due to Theorem 2.3 and the discussion before this theorem, we know that $D(x_1) = 0$, f does not belong to $k[x_1]$, and $\partial D(x_3)/\partial x_2$ does not belong to $D(x_2)k[x_1, x_2]$. Moreover, $D(x_2)$ and $D(x_3)$ are nonzero elements of $k[x_1]$ and $k[x_1, x_2]$, respectively. Hence, we can construct $h_1 \in x_2 k[x_1, x_2]$ such that $\partial h_1/\partial x_2 = -D(x_3)$ by integrating $-D(x_3)$ in x_2. Take the highest degree polynomial $f_1 \in k[x_1] \setminus \{0\}$ such that f_1 divides both $D(x_2)$ and h_1, and that $g := D(x_2)/f_1$ is a monic polynomial. Then, $h := h_1/f_1$ belongs to $x_2 k[x_1, x_2]$, and g and h have no common factor by the maximality of $\deg_{x_1} f_1$. By the definition of h, f_1 and h_1, we have

$$f_1 \frac{\partial^2 h}{\partial x_2^2} = \frac{\partial^2 (f_1 h)}{\partial x_2^2} = \frac{\partial^2 h_1}{\partial x_2^2} = -\frac{\partial D(x_3)}{\partial x_2}.$$

Since this polynomial does not belong to $D(x_2)k[x_1, x_2] = f_1 gk[x_1, x_2]$, it follows that $\partial^2 h/\partial x_2^2$ does not belong to $gk[x_1, x_2]$. Thus, (g, h) belongs to Λ. Moreover, we have $D = f_1 T_{g,h}$, since

$$D(x_1) = 0, \quad D(x_2) = f_1 g \quad \text{and} \quad D(x_3) = -\frac{\partial h_1}{\partial x_2} = -f_1 \frac{\partial h}{\partial x_2}.$$

Set $f_0 = ff_1$. Then, we get $fD = f_0 T_{g,h}$. Furthermore, f_0 belongs to $\ker D \setminus k[x_1]$, since f and f_1 belong to $k[x_1] \setminus \{0\}$ and $\ker D \setminus k[x_1]$, respectively. Because $\ker D = \ker T_{g,h}$, it remains only to show that $\ker T_{g,h} = k[x_1, gx_3 + h]$. We prove this by means of the "kernel criterion" (cf. [**9**, Proposition 5.12]). Since $gx_3 + h$ is a coordinate of $k(x_1)[x_2, x_3]$ over $k(x_1)$ such that $T_{g,h}(gx_3 + h) = 0$, we know by Theorem 2.1 that the kernel of the extension of $T_{g,h}$ to $k(x_1)[x_2, x_3]$ is generated by $gx_3 + h$ over $k(x_1)$. Hence, $\ker T_{g,h}$ is contained in $k(x_1, gx_3 + h)$. Since g and h have no common factor, and are elements of $k[x_1] \setminus \{0\}$ and $x_2 k[x_1, x_2] \setminus \{0\}$, respectively, it follows that g and $\partial h/\partial x_2$ have no common factor. Hence, $T_{g,h}(x_2)$ and $T_{g,h}(x_3)$

have no common factor. This implies that $T_{g,h}$ is irreducible, i.e., $T_{g,h}(k[\mathbf{x}])$ is contained in no proper principal ideal of $k[\mathbf{x}]$. In this situation, we may conclude that $\ker T_{g,h} = k[x_1, gx_3 + h]$ by virtue of the "kernel criterion". Therefore, f_0 belongs to $k[x_1, gx_3 + h] \setminus k[x_1]$. This proves the existence of g, h and f_0.

To prove the uniqueness, assume that $f_1 T_{g_1,h_1} = f_2 T_{g_2,h_2}$ for some $(g_i, h_i) \in \Lambda$ and $f_i \in k[x_1, g_i x_3 + h_i] \setminus k[x_1]$ for $i = 1, 2$. Then, we have

$$f_1 g_1 = f_2 g_2 \quad \text{and} \quad f_1 \frac{\partial h_1}{\partial x_2} = f_2 \frac{\partial h_2}{\partial x_2}.$$

Since g_1 and g_2 are elements of $k[x_1]$, and f_1 and f_2 are nonzero, this gives that

$$\frac{\partial g_1 h_2}{\partial x_2} = g_1 \frac{\partial h_2}{\partial x_2} = (g_1 f_2^{-1}) f_2 \frac{\partial h_2}{\partial x_2} = (g_2 f_1^{-1}) f_1 \frac{\partial h_1}{\partial x_2} = g_2 \frac{\partial h_1}{\partial x_2} = \frac{\partial g_2 h_1}{\partial x_2}.$$

Because $g_1 h_2$ and $g_2 h_1$ belong to $x_2 k[x_1, x_2]$, it follows that $g_1 h_2 = g_2 h_1$. Since g_i and h_i have no common factor for $i = 1, 2$, we may write $g_1 = c g_2$ and $h_1 = c h_2$, where $c \in k^\times$. Then, we have $c = 1$ by the assumption that g_1 and g_2 are monic polynomials. Thus, we get $g_1 = g_2$ and $h_1 = h_2$, and therefore $f_1 = f_2 g_2 / g_1 = f_2$. This proves the uniqueness of g, h and f_0. $\square$

3. Affine locally nilpotent derivations

Similarly to Problem 2, we can consider the following problem.

Problem 9. Assume that $D \in \mathrm{LND}_R R[\mathbf{x}]$ is affine. When does $\exp fD$ belong to $\mathrm{T}(R, \mathbf{x})$ for $f \in \ker D \setminus R$?

We remark that this problem is reduced to Problem 2 when $n = 2$ and $V(R) = K^\times$. Indeed, if $D \in \mathrm{LND}_R R[\mathbf{x}]$ is affine, then $\exp D$ belongs to $\mathrm{T}(R, \mathbf{x})$. Hence, D is tamely triangularizable due to the last part of Theorem 1.1.

The following is the main result of this section.

Theorem 3.1. *Assume that $n = 2$, and that $D \in \mathrm{LND}_R R[\mathbf{x}]$ is affine, and $\psi^{-1} \circ D \circ \psi$ is not triangular for any $\psi \in \mathrm{Aff}(R, \mathbf{x})$. Then,* $\exp fD$ *belongs to* $\mathrm{T}(R, \mathbf{x})$ *if and only if f belongs to R for $f \in \ker D$.*

The following is a key lemma.

Lemma 3.2. *Let D be as in Theorem 3.1. Then, the following assertions hold*:
(i) *We have* $\deg_{x_i} D(x_j) = 1$ *for every* $i, j \in \{1, 2\}$.
(ii) *There exists* $h \in R[\mathbf{x}]$ *satisfying the following conditions*:
(1) $D(h) = 0$.
(2) $1 \leq \deg_{x_1} h = \deg_{x_2} h \leq 2$.
(3) *h is a coordinate of $K[\mathbf{x}]$ over K.*
(4) *h is tamely reduced over R.*

By assuming this lemma, we can prove Theorem 3.1 as follows. The "if" part of the theorem is clear. We prove the "only if" part. Assume that $\phi := \exp fD$ belongs to $\mathrm{T}(R, \mathbf{x})$ for some $f \in \ker D$. Take $h \in R[\mathbf{x}]$ as in Lemma 3.2 (ii). Then, we have $\phi(h) = h$ by (1). Hence, ϕ belongs to $H(h)$, since ϕ belongs to $\mathrm{T}(R, \mathbf{x})$ by assumption. Because h satisfies (2), (3) and

(4), we know by Theorem 2.7 (i) that $H(h)$ is contained in $\mathrm{Aff}(R, \mathbf{x})$. Thus, ϕ belongs to $\mathrm{Aff}(R, \mathbf{x})$. By Lemma 1.5 (ii), it follows that fD is affine. By Lemma 3.2 (i), this implies that f belongs to R. Therefore, the "only if" part of Theorem 3.1 follows from Lemma 3.2.

Let us prove Lemma 3.2. Write

$$(D(x_1), D(x_2)) = (x_1, x_2)A + (b_1, b_2),$$

where $A \in M(2, R)$ and $b_1, b_2 \in R$. Then, A is a nilpotent matrix. Since D is not triangular, we have $A \neq O$. Hence, we know from linear algebra that

$$P^{-1}AP = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

for some $P \in GL(2, K)$. From this, we see that A has the form

$$A = t \begin{pmatrix} \alpha_1\alpha_2 & -\alpha_1^2 \\ \alpha_2^2 & -\alpha_1\alpha_2 \end{pmatrix}$$

for some $t \in K^\times$ and $\alpha_1, \alpha_2 \in K$. Then, α_1 is nonzero, for otherwise D is triangular if x_1 and x_2 are interchanged. Similarly, α_2 is also nonzero. Hence, we have $\deg_{x_i} D(x_j) = 1$ for every $i, j \in \{1, 2\}$. This proves Lemma 3.2 (i). We show that α_1/α_2 does not belong to $V(R)$. Suppose to the contrary that α_1/α_2 belongs to $V(R)$. Then, there exist $\alpha \in K^\times$ and $\beta_1, \beta_2 \in R$ such that $(\alpha\alpha_1)\beta_2 - (\alpha\alpha_2)\beta_1 = 1$, and $\alpha\alpha_1$ and $\alpha\alpha_2$ belong to R. Put $p = \alpha_1x_1 + \alpha_2x_2$, and define $\psi \in \mathrm{Aff}(R, \mathbf{x})$ by

$$\psi(x_1) = \alpha p, \quad \psi(x_2) = \beta_1x_1 + \beta_2x_2.$$

Then, we have $\delta := D(p) = \alpha_1b_1 + \alpha_2b_2$. Hence,

$$(\psi^{-1} \circ D \circ \psi)(x_1) = \psi^{-1}(D(\alpha p)) = \alpha\delta$$

is a constant. Since $D' := \psi^{-1} \circ D \circ \psi$ is locally nilpotent, this implies that $D'(x_2)$ belongs to $R[x_1]$. Thus, D' is triangular, a contradiction. Therefore, α_1/α_2 does not belong to $V(R)$. When $\delta = 0$, we define $h = t\alpha_2p$. Then, h belongs to $R[\mathbf{x}]$, since $t\alpha_2\alpha_i$ is an entry of A for $i = 1, 2$. Since $\delta = 0$, we get $D(h) = t\alpha_2\delta = 0$, proving (1). Since α_1 and α_2 are nonzero, we see that h is a coordinate of $K[\mathbf{x}]$ over K such that $\deg_{x_1} h = \deg_{x_2} h = 1$, proving (3) and (2). Since α_1/α_2 does not belong to $V(R)$, we know by Proposition 2.5 (ii) that h is tamely reduced over R, proving (4). Thus, h satisfies all the conditions of Lemma 3.2 (ii). Next, assume that $\delta \neq 0$. We define

$$h = t\alpha_1\delta\left(x_1 - \frac{b_1}{\delta}p - \frac{t\alpha_2}{2\delta}p^2\right).$$

Then, h belongs to $R[\mathbf{x}]$, since $t\alpha_1\delta$, $t\alpha_1p$ and $t^2\alpha_1\alpha_2p^2$ belong to $R[\mathbf{x}]$. Since

$$D(h) = t\alpha_1\delta\left((t\alpha_2p + b_1) - \frac{b_1}{\delta}\delta - \frac{t\alpha_2}{2\delta}(2p)\delta\right) = 0,$$

we get (1). It is easy to check that $\deg_{x_i} h = \deg_{x_i} p^2 = 2$ for $i = 1, 2$, proving (2). Since $K[h, p] = K[x_1, p] = K[\mathbf{x}]$, we see that h is a coordinate of $K[\mathbf{x}]$ over K, proving (3). Since $h^{\mathbf{w}(h)}$ is equal to p^2 up to a nonzero constant multiple, and since α_1/α_2 does not belong to $V(R)$, we know that h is tamely reduced over R by Proposition 2.5 (ii). Hence, h satisfies (4).

Therefore, h satisfies all the conditions of Lemma 3.2 (ii). This completes the proof of Lemma 3.2, and thereby completing the proof of Theorem 3.1.

CHAPTER 4

Tame intersection theorem

1. Main result

Assume that $n = 2$. Let S be an over domain of R, and $f \in R[\mathbf{x}]$ a coordinate of $S[\mathbf{x}]$ over S which is tamely reduced over R. As we have seen in the previous chapter, the tame intersection

$$H(f) = \mathrm{Aut}(R[\mathbf{x}]/R[f]) \cap \mathrm{T}(R, \mathbf{x})$$

plays important roles in proving the wildness of automorphisms. This motivates us to study $H(f)$ in detail.

Throughout this chapter, we assume that R contains $\mathbf{Z}$ unless otherwise stated. Under this assumption, we investigate coordinates $f \in R[\mathbf{x}]$ of $S[\mathbf{x}]$ over S which are tamely reduced over R, and for which $H(f) \neq \{\mathrm{id}_{R[\mathbf{x}]}\}$. Our goal is to give a complete classification of such f's, and to describe the concrete structures of $H(f)$'s.

We mention that $\mathrm{Aut}(R[\mathbf{x}]/R[f])$ itself is an infinite group for the following reason. Recall that, for each $D \in \mathrm{LND}_R\, R[\mathbf{x}]$, we mean by $\exp D$ the exponential automorphism for the natural extension of D to $\bar{R} := \mathbf{Q} \otimes_{\mathbf{Z}} R[\mathbf{x}]$. Since D is locally nilpotent, we may find $m \in \mathbf{N}$ such that $D^m(x_i) = 0$ for $i = 1, 2$. Then, $\phi := \exp m!D$ induces an element of $\mathrm{Aut}(R[\mathbf{x}]/R)$, since $\phi(x_i)$ and $\phi^{-1}(x_i) = (\exp -m!D)(x_i)$ belong to $R[\mathbf{x}]$ for $i = 1, 2$. Now, define $\Delta_f \in \mathrm{Der}_R\, R[\mathbf{x}]$ by

$$\Delta_f(x_1) = -\frac{\partial f}{\partial x_2} \quad \text{and} \quad \Delta_f(x_2) = \frac{\partial f}{\partial x_1}. \tag{1.1}$$

Then, we have $\Delta_f(f) = 0$. We show that Δ_f is locally nilpotent. It suffices to check that Δ_f extends to a locally nilpotent derivation of $S[\mathbf{x}]$. Since f is a coordinate of $S[\mathbf{x}]$ over S, there exists $\psi \in \mathrm{Aut}(S[\mathbf{x}]/S)$ such that $\psi(x_1) = f$. Then, we have $\Delta_f(\psi(x_1)) = \Delta_f(f) = 0$, and

$$\Delta_f(\psi(x_2)) = -\frac{\partial f}{\partial x_2}\frac{\partial \psi(x_2)}{\partial x_1} + \frac{\partial f}{\partial x_1}\frac{\partial \psi(x_2)}{\partial x_2} = \det J\psi.$$

Since $\det J\psi$ belongs to $S^\times$, it follows that $\Delta_f^2(\psi(x_2)) = 0$. Hence, Δ_f extends to a locally nilpotent derivation of $S[\mathbf{x}] = S[\psi(x_1), \psi(x_2)]$. Thus, Δ_f is locally nilpotent. As remarked, there exists $c \in \mathbf{N}$ such that $\phi := \exp c\Delta_f$ induces an element of $\mathrm{Aut}(R[\mathbf{x}]/R)$. Then, ϕ belongs to $\mathrm{Aut}(R[\mathbf{x}]/R[f])$, since $\Delta_f(f) = 0$. Because ϕ has an infinite order, we conclude that $\mathrm{Aut}(R[\mathbf{x}]/R[f])$ is an infinite group

Now, let K be the field of fractions of R. Then, K is of characteristic zero by the assumption that R contains $\mathbf{Z}$. Thanks to the following lemma, we may assume that $S = K$ without loss of generality.

Lemma 1.1. *If $f \in R[\mathbf{x}]$ is a coordinate of $S[\mathbf{x}]$ over S, then f is a coordinate of $K[\mathbf{x}]$ over K.*

PROOF. Let U be the set of $f \in K[\mathbf{x}]$ such that f is not a coordinate of $K[\mathbf{x}]$ over K, but is a coordinate of $L[\mathbf{x}]$ over L, where L is the field of fractions of S. Then, $\tau(U)$ is contained in U for each $\tau \in \mathrm{Aut}(K[\mathbf{x}]/K)$. Now, suppose to the contrary that there exists a coordinate $f' \in R[\mathbf{x}]$ of $S[\mathbf{x}]$ over S which is not a coordinate of $K[\mathbf{x}]$ over K. Then, f' is a coordinate of $L[\mathbf{x}]$ over L. Hence, f' belongs to U. Thus, U is not empty. Take $f \in U$ so that $|\mathbf{w}(f)|$ is minimal. Then, $\tau(f)$ belongs to U for each $\tau \in \mathrm{Aut}(K[\mathbf{x}]/K)$. Hence, we get $|\mathbf{w}(\tau(f))| \geq |\mathbf{w}(f)|$ by the minimality of $|\mathbf{w}(f)|$. Thus, f is tamely reduced over K. Since f is not a coordinate of $K[\mathbf{x}]$ over K, we see that f is not a linear polynomial. Because f is not a constant, we get $|\mathbf{w}(f)| > 1$. By applying Proposition 1.2 with $\kappa = L$, we may write $f^{\mathbf{w}(f)} = a(x_i + bx_j^l)^m$, where $a, b \in L^\times$, $i, j \in \{1, 2\}$ with $i \neq j$ and $l, m \in \mathbf{N}$. Then, ax_i^m and $mabx_i^{m-1}x_j^l$ belong to $K[\mathbf{x}]$, since f is an element of $K[\mathbf{x}]$. Hence, a belongs to $K^\times$. Since $m \geq 1$, and K is of characteristic zero, it follows that b belongs to $K^\times$. Thus, b belongs to K and $V(K)$. Thanks to Proposition 2.5, this implies that f is not tamely reduced over K, a contradiction. Therefore, every coordinate $f \in R[\mathbf{x}]$ of $S[\mathbf{x}]$ over S is a coordinate of $K[\mathbf{x}]$ over K. □

When R does not contain $\mathbf{Z}$, a statement similar to Lemma 1.1 does not hold in general. We will give a counterexample at the end of the next section.

Let us define five types of elements of $R[\mathbf{x}]$. In (1) and (2) of the following definition, $c \in R \setminus \{0\}$ denotes the leading coefficient of the polynomial $g \in R[x_1]$. It is easy to check that the following five types of polynomials are coordinates of $K[\mathbf{x}]$ over K.

Definition 1.1. Let f be an element of $R[\mathbf{x}]$.
(1) We say that f is of *type I* if

$$f = f_1 := ax_2 + g$$

for some $a \in R \setminus \{0\}$ and $g \in R[x_1]$ such that $\deg_{x_1} g \geq 2$, and c does not belong to aR.
(2) We say that f is of *type II* if

$$f = f_2 := a'x_1 + h$$

for some $a' \in K^\times$ and $h \in K[\mathbf{x}]$ as follows:
(a) There exist $\zeta \in R \setminus \{1\}$ and $e \geq 2$ such that $\zeta^e = 1$ and a' belongs to $R' := R[(\zeta - 1)^{-1}]$.
(b) There exists $g \in R[x_1]$ such that $\deg_{x_1} g \geq 2$, c does not belong to $(\zeta - 1)R$ and h belongs $R'[y_2^e] \setminus R'$, where $y_2 := (\zeta - 1)x_2 + g$.
(3) We say that f is of *type III* if

$$f = f_3 := a_1x_1 + a_2x_2 + b$$

for some $a_1, a_2 \in R \setminus \{0\}$ and $b \in R$ such that a_1/a_2 does not belong to $V(R)$.

(4) We say that f is of *type IV* if

$$f = f_4 := a\tau_4(x_1)^2 + \tau_4(x_2)$$

for some $a \in R \setminus \{0\}$, and $\tau_4 \in \mathrm{Aff}(K, \mathbf{x})$ for which there exist $\alpha_1, \alpha_2 \in K^\times$ such that $\tau_4(x_1) = \alpha_1 x_1 + \alpha_2 x_2$ and α_1/α_2 does not belong to $V(R)$.
(5) We say that f is of *type V* if

$$f = f_5 := \tau_5(x_2 + g')$$

for some $\tau_5 \in \mathrm{Aff}(K, \mathbf{x})$ and $g' \in K[x_1] \setminus \{0\}$ with $\deg_{x_1} g' \geq 3$ as follows:
(a) g' belongs to $x_1^{e'} K[x_1^{e'}]$ for some $e' \geq 2$.
(b) Let $\alpha_i, \beta_i \in K$ for $i = 0, 1, 2$ be such that

$$\tau_5(x_1) = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_0 \quad \text{and} \quad \tau_5(x_2) = \beta_1 x_1 + \beta_2 x_2 + \beta_0.$$

Then, we have $\alpha_i \neq 0$ for $i = 1, 2$, and α_1/α_2 does not belong to $V(R)$.
(c) There exists $\zeta' \in R \setminus \{1\}$ such that $(\zeta')^{e'} = 1$, and

$$\gamma_{i,j}(\zeta') := \frac{(\zeta' - 1)\alpha_i \beta_j}{\alpha_1 \beta_2 - \alpha_2 \beta_1}$$

belongs to R for $i = 0, 1, 2$ and $j = 1, 2$.

By definition, no element of $R[\mathbf{x}]$ is of type III or IV or V if $V(R) = K^\times$. If R is a $\mathbf{Q}$-domain and $\zeta \in R \setminus \{1\}$ is a root of unity, then $\zeta - 1$ is a nonzero element of $\mathbf{Q}[\zeta] = \mathbf{Q}(\zeta)$, and hence belongs to $R^\times$. Thus, no element of $R[\mathbf{x}]$ is of type II by (b) of (2). If R is a $\mathbf{Q}$-domain, then (b) and (c) of (5) imply that $\beta_i \neq 0$ for $i = 1, 2$. In fact, if $\beta_1 = 0$, then we have $\gamma_{2,2}(\zeta') = (\zeta' - 1)\alpha_2/\alpha_1$. Since $\zeta' - 1$ belongs to $R^\times$, it follows from (c) that α_2/α_1 belongs to R. Hence, α_1/α_2 belongs to $V(R)$, a contradiction to (b). Similarly, we get a contradiction if $\beta_2 = 0$, since $\gamma_{1,1}(\zeta') = (1 - \zeta')\alpha_1/\alpha_2$ belongs to R.

Put $\lambda = \deg_{x_1} g$, $\lambda_1 = \deg_{y_2} h$ and $\lambda_2 = \deg_{x_1} g'$, where we regard h as a polynomial in y_2 over R'. Then, we have $\lambda \geq 2$, $\lambda_1 \geq 2$, $\lambda_2 \geq 3$ and

$$\mathbf{w}(f_i) = (\deg_{x_2} f_i, \deg_{x_1} f_i) = \begin{cases} (1, \lambda) & \text{if } i = 1 \\ (\lambda_1, \lambda\lambda_1) & \text{if } i = 2 \\ (1, 1) & \text{if } i = 3 \\ (2, 2) & \text{if } i = 4 \\ (\lambda_2, \lambda_2) & \text{if } i = 5. \end{cases} \tag{1.2}$$

Let $c_1 \in R'$ and $c_2 \in K$ be the leading coefficients of h and g', respectively. Then, we have

$$f_i^{\mathbf{w}(f_i)} = \begin{cases} a x_2 + c x_1^\lambda & \text{if } i = 1 \\ c_1\big((\zeta - 1)x_2 + c x_1^\lambda\big)^{\lambda_1} & \text{if } i = 2 \\ a_1 x_1 + a_2 x_2 & \text{if } i = 3 \\ a(\alpha_1 x_1 + \alpha_2 x_2)^2 & \text{if } i = 4 \\ c_2(\alpha_1 x_1 + \alpha_2 x_2)^{\lambda_2} & \text{if } i = 5. \end{cases} \tag{1.3}$$

By assumption, c/a does not belong to R when $i = 1$, $c/(\zeta - 1)$ does not belong to R when $i = 2$, a_1/a_2 does not belong to $V(R)$ when $i = 3$, and α_1/α_2 does not belong to $V(R)$ when $i = 4, 5$. Hence, we know by Proposition 2.5 that f_i is tamely reduced over R for $i = 1, \ldots, 5$. This implies that, if f_i is a coordinate of $R[\mathbf{x}]$ over R, then f_i is wild. Indeed, if

a tame coordinate of $R[\mathbf{x}]$ over R is tamely reduced over R, then it must be a linear polynomial in x_i over R for some $i \in \{1, 2\}$.

The following is the main theorem of this chapter.

Theorem 1.2. *Assume that $n = 2$. Let R be a domain containing $\mathbf{Z}$, K the field of fractions of R, and $f \in R[\mathbf{x}]$ a coordinate of $K[\mathbf{x}]$ over K. Then, f is of one of the types I through V if and only if the following three conditions hold*:
(A) *$H(f)$ is not equal to $\{\mathrm{id}_{R[\mathbf{x}]}\}$.*
(B) *f is tamely reduced over R.*
(C) $\deg_{x_1} f \geq \deg_{x_2} f \geq 1$.

As discussed in the next section, there exist many elements of $R[\mathbf{x}]$ which are coordinates of $K[\mathbf{x}]$ over K satisfying (B) and (C), but are of none of the types I through V. For such $f \in R[\mathbf{x}]$, we have $H(f) = \{\mathrm{id}_{R[\mathbf{x}]}\}$ due to Theorem 1.2. Since $\mathrm{Aut}(R[\mathbf{x}]/R[f])$ itself is an infinite group, this means the existence of a large number of wild automorphisms.

When R does not contain $\mathbf{Z}$, a statement similar to the "if" part of Theorem 1.2 does not hold in general. We will give a counterexample at the end of the next section.

Next, we define a subset H_i of $\mathrm{Aut}(R[\mathbf{x}]/R)$ for the polynomial f_i in Definition 1.1 for $i = 1, \ldots, 5$.

Definition 1.2. (1) For f_1, we define

$$H_1 = \{\phi \in J(R; x_1, x_2) \mid \phi(x_2) = x_2 + a^{-1}(g - \phi(g))\}.$$

(2) For f_2, let μ be the maximal integer such that h belongs to $R'[y_2^\mu]$, and Z the set of $\omega \in R^\times$ such that $\omega^\mu = 1$ and $g_\omega := (\omega - 1)(\zeta - 1)^{-1} g$ belongs to $R[x_1]$. Then, we define

$$H_2 = \{\phi \in \mathrm{Aut}(R[\mathbf{x}]/R[x_1]) \mid \phi(x_2) = \omega x_2 + g_\omega \text{ for some } \omega \in Z\}.$$

(3) For f_3, we define H_3 to be the set of $\phi \in \mathrm{Aff}(R, \mathbf{x})$ defined by

$$(\phi(x_1), \phi(x_2)) = (x_1, x_2)A + (b_1, b_2) \tag{1.4}$$

for some $A \in GL(2, R)$ and $b_1, b_2 \in R$ such that

$$A\mathbf{a} = \mathbf{a} \quad \text{and} \quad a_1 b_1 + a_2 b_2 = 0, \quad \text{where} \quad \mathbf{a} := \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}. \tag{1.5}$$

(4) For f_4, we define H_4 to be the set of $\phi \in \mathrm{Aff}(R, \mathbf{x})$ such that

$$\big((\tau_4^{-1} \circ \phi \circ \tau_4)(x_1), (\tau_4^{-1} \circ \phi \circ \tau_4)(x_2)\big) = (x_1, x_2) \begin{pmatrix} \epsilon & 2\epsilon\alpha a \\ 0 & 1 \end{pmatrix} - (\alpha, \alpha^2 a) \tag{1.6}$$

for some $\epsilon \in \{1, -1\}$ and $\alpha \in K$.
(5) For f_5, let μ' be the maximal integer such that g' belongs to $K[x_1^{\mu'}]$, and Z' the set of $\omega \in R^\times$ such that $\omega^{\mu'} = 1$ and $\gamma_{i,j}(\omega)$ belongs to R for $i = 0, 1, 2$ and $j = 1, 2$. Then, we can define $\phi_\omega \in \mathrm{Aff}(R, \mathbf{x})$ by

$$\begin{aligned} \phi_\omega(x_1) &= (1 + \gamma_{1,2}(\omega))x_1 + \gamma_{2,2}(\omega)x_2 + \gamma_{0,2}(\omega) \\ \phi_\omega(x_2) &= -\gamma_{1,1}(\omega)x_1 + (1 - \gamma_{2,1}(\omega))x_2 - \gamma_{0,1}(\omega) \end{aligned} \tag{1.7}$$

for each $\omega \in Z'$. Actually, we have $\det J\phi_\omega = \omega$, since

$$(1+\gamma_{1,2}(\omega))(1-\gamma_{2,1}(\omega)) - (-\gamma_{1,1}(\omega))\gamma_{2,2}(\omega)$$
$$= 1 + \frac{(\omega-1)(\alpha_1\beta_2-\alpha_2\beta_1)}{\alpha_1\beta_2-\alpha_2\beta_1} - \frac{(\omega-1)^2\alpha_1\beta_2\alpha_2\beta_1}{(\alpha_1\beta_2-\alpha_2\beta_1)^2} + \frac{(\omega-1)^2\alpha_1\beta_1\alpha_2\beta_2}{(\alpha_1\beta_2-\alpha_2\beta_1)^2} = \omega.$$

We define $H_5 = \{\phi_\omega \mid \omega \in Z'\}$.

In the notation above, the following theorem holds.

Theorem 1.3. *Let $f_1, \ldots, f_5$ be as in Definition 1.1, where f_5 need not to satisfy* (c) *of* (5). *Then, we have $H(f_i) = H_i$ for $i = 1, \ldots, 5$.*

We remark that H_2 is a finite set with at most μ elements, since Z consists of μ-th roots of unity. Similarly, H_5 is a finite set with at most μ' elements. Thus, $H(f_2)$ and $H(f_5)$ are finite groups due to Theorem 1.3. Therefore, if f is of type II or V for $f \in R[\mathbf{x}]$, then f is quasi-totally wild.

Proposition 1.4. *f_1, f_3 and f_4 are not exponentially wild. In particular, $H(f_1)$, $H(f_3)$ and $H(f_4)$ are infinite groups.*

Proof. For $i = 1, 3, 4$, define Δ_{f_i} as in (1.1). Then, Δ_{f_i} is locally nilpotent, since f_i is a coordinate of $K[\mathbf{x}]$ over K. Take $m_i \in \mathbf{N}$ such that $\Delta_{f_i}^{m_i}(x_j) = 0$ for $j = 1, 2$, and set $D_i = m_i!\Delta_{f_i}$. Then, we have $D_i(f_i) = 0$, and $\exp D_i$ induces an element of $\mathrm{Aut}(R[\mathbf{x}]/R)$. From the definition of f_1 and f_3, we see that $\partial f_i/\partial x_2$ belongs to R, and $\partial f_i/\partial x_1$ belongs to $R[x_1]$ for $i = 1, 3$. Hence, D_1 and D_3 are triangular. Thus, $\exp D_1$ and $\exp D_3$ belong to $\mathrm{T}(R, \mathbf{x})$. Since $\partial f_4/\partial x_i$ is a linear polynomial for $i = 1, 2$, we see that D_4 is affine. Hence, $\exp D_4$ belongs to $\mathrm{T}(R, \mathbf{x})$. Therefore, f_i is not exponentially wild for $i = 1, 3, 4$. Since $\exp D_i$ belongs to $H(f_i)$ and has an infinite order, we know that $H(f_i)$ is an infinite group for $i = 1, 3, 4$. $\square$

As a consequence of Theorems 1.2 and 1.3 and Proposition 1.4, we get the following corollary.

Corollary 1.5. *Assume that $f \in R[\mathbf{x}]$ is a coordinate of $K[\mathbf{x}]$ over K.*
(i) *f is quasi-totally wild if and only if f is exponentially wild.*
(ii) *Assume that R is a $\mathbf{Q}$-domain such that $V(R) = K^\times$. If $H(f)$ is not equal to $\{\mathrm{id}_{R[\mathbf{x}]}\}$, then $H(f)$ is an infinite group.*

Proof. (i) It suffices to prove the "if" part. Assume that f is exponentially wild. Without loss of generality, we may assume that $H(f) \neq \{\mathrm{id}_{R[\mathbf{x}]}\}$. By replacing f with $\tau(f)$ for some $\tau \in \mathrm{T}(R, \mathbf{x})$, we may assume further that f is tamely reduced over R and $\deg_{x_1} f \geq \deg_{x_2} f$. Since f is exponentially wild by assumption, f is not killed by $\partial/\partial x_2$. Hence, we get $\deg_{x_2} f \geq 1$. Thus, f satisfies (A), (B) and (C) of Theorem 1.2. Therefore, f must be of one of the types I through V. By Proposition 1.4, f is not of type I or III or IV. Hence, f must be of type II or V. Therefore, f is quasi-totally wild as mentioned after Theorem 1.3.

(ii) By replacing f with $\tau(f)$ for some $\tau \in \mathrm{T}(R, \mathbf{x})$, we may assume that f is tamely reduced over R and $\deg_{x_1} f \geq \deg_{x_2} f$. If $\deg_{x_2} f = 0$, then f is a linear polynomial in x_1 over R. Hence, we have $H(f) = H(x_1)$. Thus, $H(f)$ is an infinite group. Assume that $\deg_{x_2} f \geq 1$, and $H(f) \neq \{\mathrm{id}_{R[\mathbf{x}]}\}$. Then, f satisfies (A), (B) and (C) of Theorem 1.2. Hence, f must be of one

of the types I through V. Since R is a $\mathbf{Q}$-domain such that $V(R) = K^\times$, we know that f is of none of the types II through V by the remark after Definition 1.1. Thus, f is of type I. Therefore, $H(f)$ is an infinite group by Proposition 1.4. □

Thanks to Theorem 1, Corollary 1.5 (ii) implies the following corollary.

Corollary 1.6. *Assume that $n = 3$. Let $f \in k[\mathbf{x}]$ be a coordinate of $k(x_3)[x_1, x_2]$ over $k(x_3)$. Then, $\mathrm{Aut}(k[\mathbf{x}]/k[x_3, f]) \cap \mathrm{T}(k, \mathbf{x})$ is equal to $\{\mathrm{id}_{k[\mathbf{x}]}\}$ or an infinite group.*

Actually, $R = k[x_3]$ is a $\mathbf{Q}$-domain such that $V(R) = K^\times$, and

$$H(f) = \mathrm{Aut}(k[\mathbf{x}]/k[x_3, f]) \cap \mathrm{T}(k[x_3], \{x_1, x_2\}) = \mathrm{Aut}(k[\mathbf{x}]/k[x_3, f]) \cap \mathrm{T}(k, \mathbf{x})$$

by Theorem 1.

2. Examples

In this section, we construct various examples. First, we give the five types of polynomials. It is easy to find polynomials of types I through IV. For example, let $R = \mathbf{Z}$. Then, $g_1 := 2x_2 + x_1^2$ is of type I, and

$$g_2 := x_1 + (2x_2 + x_1^2)^2$$

is of type II with $\zeta = -1$, $e = 2$ and $g = -x_1^2$. Let $R = \mathbf{Z}[a_1, a_2]$ be the polynomial ring in a_1 and a_2 over $\mathbf{Z}$. Then, a_1/a_2 does not belong to $V(R)$ by Lemma 3.1 (ii). Hence, $g_3 := a_1x_1 + a_2x_2$ is of type III, and

$$g_4 := (a_1x_1 + a_2x_2)^2 + x_2$$

is of types IV with $a = 1$ and $\tau_4 \in \mathrm{Aff}(K, \mathbf{x})$ defined by $\tau_4(x_1) = a_1x_1 + a_2x_2$ and $\tau_4(x_2) = x_2$.

To construct a polynomial of type V, consider the subring $R := \mathbf{Z}[y, 2z, yz]$ of the polynomial ring $\mathbf{Z}[y, z]$. Then, it is easy to see that y is an irreducible element of R, z does not belong to R, and $I := yR + yzR$ is not equal to R. We claim that I is not a principal ideal of R. In fact, if $I = pR$ for some $p \in R \setminus R^\times$, then p divides y, since y belongs to I. Hence, we get $pR = yR$ by the irreducibility of y. Since yz belongs to I, it follows that y divides yz. Hence, z belongs to R, a contradiction. Thus, I is not a principal ideal of R. Therefore, yz/y does not belong to $V(R)$ by Lemma 3.1 (ii). Define $\tau_5 \in \mathrm{Aff}(K, \mathbf{x})$ by $\tau_5(x_1) = yx_1 + yzx_2$ and $\tau_5(x_2) = x_2$. Then,

$$g_5 := \tau_5(x_2 + x_1^4)$$

is an element of $R[\mathbf{x}]$ of the form of f_5 with $g' = x_1^4$, and τ_5 satisfies (b) of Definition 1.1 (5). Since x_1^4 belongs to $x_1^4K[x_1^4]$, we see that (a) holds for $e' = 4$. Observe that $\gamma_{1,2}(-1) = -2y/y = -2$, $\gamma_{2,2}(-1) = -2yz/y = -2z$ and $\gamma_{i,j}(-1) = 0$ if $i = 0$ or $j = 1$. Hence, $\gamma_{i,j}(-1)$ belongs to R for $i = 0, 1, 2$ and $j = 1, 2$. Since $(-1)^4 = 1$, we conclude that (c) holds for $\zeta = -1$. Therefore, g_5 is of type V.

We remark that $g_1, \ldots, g_5$ above are not coordinates of $R[\mathbf{x}]$ over R. This can be verified by using the following lemma and proposition.

Lemma 2.1. *Let f be a coordinate of $R[\mathbf{x}]$ over R, $\mathfrak{p}$ a prime ideal of R, and $\bar{f}$ the image of f in $(R/\mathfrak{p})[\mathbf{x}]$. If $\bar{f}$ belongs to $(R/\mathfrak{p})[x_i]$ for some $i \in \{1, 2\}$, then we have $\deg_{x_i} f = 1$.*

PROOF. Since f is a coordinate of $R[\mathbf{x}]$ over R, we may find $\phi \in \mathrm{Aut}(R[\mathbf{x}]/R)$ such that $\phi(x_i) = f$. Set $\bar{R} = R/\mathfrak{p}$. Then, $\bar{\phi} := \mathrm{id}_{\bar{R}} \otimes \phi$ is an automorphism of $\bar{R}[\mathbf{x}] = \bar{R} \otimes_R R[\mathbf{x}]$ over $\bar{R}$. Since $\bar{\phi}(x_i) = \bar{f}$ belongs to $\bar{R}[x_i]$ by assumption, $\bar{\phi}$ induces an element of $\mathrm{Aut}(\bar{R}[x_i]/\bar{R})$. Because $\bar{R}$ is a domain, this implies that $\bar{\phi}(x_i)$ is a linear polynomial. Therefore, we get $\deg_{x_i} \bar{f} = 1$. □

Since the images of g_1 and g_2 in $(\mathbf{Z}/2\mathbf{Z})[\mathbf{x}]$ are x_1^2 and $x_1 + x_1^4$, respectively, we know by Lemma 2.1 that g_1 and g_2 are not coordinates of $\mathbf{Z}[\mathbf{x}]$ over $\mathbf{Z}$. Since the image of g_4 in $(\mathbf{Z}[a_1, a_2]/(a_1))[\mathbf{x}]$ is $(a_2x_2)^2 + x_2$, we know that g_4 is not a coordinate of $(\mathbf{Z}[a_1, a_2])[\mathbf{x}]$ over $\mathbf{Z}[a_1, a_2]$ similarly. As for g_5, observe that $\mathfrak{p} = (y, 2z, 2)$ is a prime ideal of $R = \mathbf{Z}[y, 2z, yz]$. Since the image of g_5 in $(R/\mathfrak{p})[\mathbf{x}]$ is $(yzx_2)^4 + x_2$, we know that g_5 is not a coordinate of $R[\mathbf{x}]$ over R.

By the following proposition, g_3 is not a coordinate of $R[\mathbf{x}]$ over R.

Proposition 2.2. *No coordinate of $R[\mathbf{x}]$ over R is of type III.*

PROOF. Suppose to the contrary that $\phi(x_1) = f_3$ for some $\phi \in \mathrm{Aut}(R[\mathbf{x}]/R)$. Then, $\det J\phi$ belongs to $a_1R + a_2R$, since $\partial f_3/\partial x_i = a_i$ for $i = 1, 2$. Because $\det J\phi$ belongs to $R^\times$, we get $a_1R + a_2R = R$. This contradicts that a_1/a_2 does not belong to $V(R)$. Therefore, f_3 is not a coordinate of $R[\mathbf{x}]$ over R. □

Next, we give examples of coordinates of $R[\mathbf{x}]$ over R which are of types I, IV and V. The following fact is well-known (see [**6**, Lemma 1.1.8] for a more general statement).

Lemma 2.3. *Let ϕ be an endomorphism of the R-algebra $R[\mathbf{x}]$ such that $\det J\phi$ belongs to $R^\times$. If $K[\phi(x_1), \phi(x_2)] = K[\mathbf{x}]$, then ϕ belongs to $\mathrm{Aut}(R[\mathbf{x}]/R)$.*

Assume that $R = \mathbf{Z}$. Then, it is easy to see that

$$h_1 := 4x_2 + 1 + x_1 + 2x_1^2$$

is a type I element of $R[\mathbf{x}]$. We show that h_1 is a coordinate of $R[\mathbf{x}]$ over R. Since

$$2h_1^2 - 3h_1 \equiv 2(1 + x_1)^2 + (1 + x_1 + 2x_1^2) \equiv x_1 - 1 \pmod{4R[\mathbf{x}]},$$

we see that

$$h_1' := \frac{1}{4}\left(x_1 - 1 - (2h_1^2 - 3h_1)\right)$$

belongs to $R[\mathbf{x}]$. Hence, we may define an endomorphism ϕ of the R-algebra $R[\mathbf{x}]$ by $\phi(x_1) = h_1$ and $\phi(x_2) = h_1'$. Then, we have $\det J\phi = -1$, since

$$dh_1 \wedge dh_1' = \frac{1}{4} dh_1 \wedge dx_1 = -dx_1 \wedge dx_2.$$

Thus, $\det J\phi$ belongs to $R^\times$. Since $K[h_1, h_1'] = K[h_1, x_1] = K[\mathbf{x}]$, we conclude that ϕ belongs to $\mathrm{Aut}(R[\mathbf{x}]/R)$ by Lemma 2.3. Therefore, h_1 is a coordinate of $R[\mathbf{x}]$ over R.

Next, let $R = \mathbf{Z}[\alpha_1, \alpha_2]$ be the polynomial ring in α_1 and α_2 over $\mathbf{Z}$. Define elements of $R[\mathbf{x}]$ by

$$h_4 = \alpha_1(\alpha_1 x_1 + \alpha_2 x_2)^2 + x_2, \quad h_4' = \alpha_2(\alpha_1 x_1 + \alpha_2 x_2)^2 - x_1.$$

Then, h_4 is of type IV with $a = \alpha_1$ and $\tau_4 \in \mathrm{Aff}(K, \mathbf{x})$ defined by $\tau_4(x_1) = \alpha_1 x_1 + \alpha_2 x_2$ and $\tau_4(x_2) = x_2$. Since $\alpha_2 h_4 - \alpha_1 h_4' = \alpha_1 x_1 + \alpha_2 x_2$, we see that x_1 and x_2 belong to $R[h_4, h_4']$. Hence, we get $R[h_4, h_4'] = R[\mathbf{x}]$. Therefore, h_4 is a coordinate of $R[\mathbf{x}]$ over R.

Finally, we construct a coordinate of type V. Let R_0 be the polynomial ring in four variables α_i and β_i for $i = 1, 2$ over $\mathbf{Z}$. Put $\delta = \alpha_1\beta_2 - \alpha_2\beta_1$, and consider the subring

$$R := R_0\left[2\alpha_1\delta^{-1}, 2\alpha_2\delta^{-1}, 4\beta_1\delta^{-1}, 4\beta_2\delta^{-1}, (\alpha_1 + 2\beta_1)\delta^{-1}, (\alpha_2 + 2\beta_2)\delta^{-1}\right]$$

of $R_0[\delta^{-1}]$. Define $\tau_5 \in \mathrm{Aff}(K, \mathbf{x})$ by

$$\tau_5(x_1) = \alpha_1 x_1 + \alpha_2 x_2 + 2 \quad \text{and} \quad \tau_5(x_2) = \beta_1 x_1 + \beta_2 x_2.$$

Then, $h_5 := \tau_5(x_2 + x_1^4)$ is an element of $R[\mathbf{x}]$ of the form of f_5 with $g' = x_1^4$, and satisfies (a) of Definition 1.1 (5) for $e' = 4$. We check (b) and (c).

Define a homomorphism $\psi : R_0 \to \mathbf{Z}$ of $\mathbf{Z}$-algebras by

$$\psi(\alpha_1) = 0, \quad \psi(\alpha_2) = 2, \quad \psi(\beta_1) = -2, \quad \psi(\beta_2) = 3.$$

Then, we have $\psi(\delta) = 4$. Hence, ψ extends to a homomorphism $\bar{\psi} : R_0[\delta^{-1}] \to \mathbf{Z}[1/4]$ of $\mathbf{Z}$-algebras. Then, we have $\bar{\psi}\big(4\beta_i\delta^{-1}\big) = \psi(\beta_i)$ for $i = 1, 2$, and

$$\bar{\psi}\big(2\alpha_1\delta^{-1}\big) = 0,\ \bar{\psi}\big(2\alpha_2\delta^{-1}\big) = 1,\ \bar{\psi}\big((\alpha_1{+}2\beta_1)\delta^{-1}\big) = -1,\ \bar{\psi}\big((\alpha_2{+}2\beta_2)\delta^{-1}\big) = 2.$$

Thus, $\bar{\psi}(R)$ is contained in $\mathbf{Z}$. Therefore, we get $\bar{\psi}(R) = \mathbf{Z}$.

We prove that 2 is an irreducible element of R by contradiction. Suppose that $2 = pq$ for some $p, q \in R \setminus \{0\}$ not belonging to $R^\times$. Since p and q belong to $R_0[\delta^{-1}]$, we may write $p = p'\delta^{-l}$ and $q = q'\delta^{-m}$, where $p', q' \in R_0$ and $l, m \in \mathbf{Z}_{\geq 0}$. Then, we have $2\delta^{l+m} = p'q'$. Since 2 and δ are irreducible elements of R_0, we may write $p' = 2u\delta^{l'}$ and $q' = u\delta^{m'}$ by interchanging p' and q' if necessary, where $u \in \{1, -1\}$, and $l', m' \in \mathbf{Z}_{\geq 0}$ are such that $l' + m' = l + m$. We show that $(l, m) = (l', m')$. Then, it follows that $q = (u\delta^{m'})\delta^{-m} = u$ belongs to $R^\times$, and we are led to a contradiction. Suppose to the contrary that $(l, m) \neq (l', m')$. Then, we have $l > l'$ or $m > m'$. If $l > l'$, then $u\delta^{l-l'-1}p$ belongs to R, since so does p. Since $p = (2u\delta^{l'})\delta^{-l}$, we have $u\delta^{l-l'-1}p = 2\delta^{-1}$. Hence, $2\delta^{-1}$ belongs to R. Thus, $\bar{\psi}(2\delta^{-1}) = 1/2$ belongs to $\bar{\psi}(R) = \mathbf{Z}$, a contradiction. Similarly, if $m > m'$, then $u\delta^{m-m'-1}q$ belongs to R. Since $q = (u\delta^{m'})\delta^{-m}$, we have $u\delta^{m-m'-1}q = \delta^{-1}$. Hence, δ^{-1} belongs to R. Thus, $\bar{\psi}(\delta^{-1}) = 1/4$ belongs to $\bar{\psi}(R) = \mathbf{Z}$, a contradiction. Therefore, 2 is an irreducible element of R.

We show that $I := \alpha_1 R + \alpha_2 R$ is not a principal ideal of R. First, note that I is not a unit ideal, since $\bar{\psi}(I) = 2\mathbf{Z}$. Suppose that $I = sR$ for some $s \in R \setminus R^\times$. Then, s divides

$$\alpha_1\left((\alpha_2 + 2\beta_2)\delta^{-1}\right) - \alpha_2\left((\alpha_1 + 2\beta_1)\delta^{-1}\right) = 2(\alpha_1\beta_2 - \alpha_2\beta_1)\delta^{-1} = 2.$$

Since s is not a unit of R, it follows that $sR = 2R$ by the irreducibility of 2. Hence, 2 divides α_1. Thus, $\alpha_1/2$ belongs to R, and so belongs to $R_0[\delta^{-1}]$, a contradiction. Therefore, I is not a principal ideal of R. Consequently, α_1/α_2 does not belong to $V(R)$ by Lemma 3.1 (ii). This proves that τ_5 satisfies (b) of Definition 1.1 (5). Observe that

$$\gamma_{i,j}(-1) = -2\alpha_i\beta_j\delta^{-1} = -\beta_j(2\alpha_i\delta^{-1}) \quad \text{and} \quad \gamma_{0,j}(-1) = -4\beta_j\delta^{-1}$$

belong to R for each $i, j \in \{1, 2\}$. Since $(-1)^{e'} = 1$, we know that (c) holds for $\zeta = -1$. Therefore, h_5 is of type V.

Next, we show that h_5 is a coordinate of $R[\mathbf{x}]$ over R. Consider the polynomial

$$h_5' := \delta^{-1}\bigl(\tau_5(x_1) + 2(h_5 - 17)\bigr).$$

Then, we have $K[h_5, h_5'] = K[h_5, \tau_5(x_1)] = K[\tau_5(x_2), \tau_5(x_1)] = K[\mathbf{x}]$, and

$$dh_5 \wedge dh_5' = \delta^{-1} dh_5 \wedge d\tau_5(x_1) = \delta^{-1} d\tau_5(x_2) \wedge d\tau_5(x_1) = -dx_1 \wedge dx_2. \tag{2.1}$$

We check that h_5' belongs to $R[\mathbf{x}]$. Note that $p := \tau_5(x_1)^4 - 16$ is divisible by $\alpha_1 x_1 + \alpha_2 x_2$. Since $2\delta^{-1}(\alpha_1 x_1 + \alpha_2 x_2)$ belongs to $R[\mathbf{x}]$, we see that $2\delta^{-1} p$ belongs to $R[\mathbf{x}]$. Since $h_5 - 17 = \tau(x_2) + p - 1$, it follows that

$$h_5' = \delta^{-1}\bigl(\tau_5(x_1) + 2(\tau(x_2) + p - 1)\bigr) = (\alpha_1 + 2\beta_1)\delta^{-1} x_1 + (\alpha_2 + 2\beta_2)\delta^{-1} x_2 + 2\delta^{-1} p$$

belongs to $R[\mathbf{x}]$. Hence, we may define an endomorphism ϕ of the R-algebra $R[\mathbf{x}]$ by $\phi(x_1) = h_5$ and $\phi(x_2) = h_5'$. Then, we have $\det J\phi = -1$ by (2.1). Hence, $\det J\phi$ belongs to $R^\times$. Thus, ϕ belongs to $\mathrm{Aut}(R[\mathbf{x}]/R)$ by Lemma 2.3. Therefore, h_5 is a coordinate of $R[\mathbf{x}]$ over R.

As remarked after Theorem 1.2, we can easily construct elements of $R[\mathbf{x}]$ which are coordinates of $K[\mathbf{x}]$ over K satisfying (B) and (C) of Theorem 1.2, but are of none of the types I through V. For example, take any polynomial f_1 of type I, and $\Phi_i(z) \in R[z]$ with $e_i := \deg_z \Phi_i(z) \geq 2$ for each $i \geq 1$. We define a sequence $(p_i)_{i=0}^\infty$ of elements of $R[\mathbf{x}]$ by

$$p_0 = x_1, \quad p_1 = f_1 \quad \text{and} \quad p_{i+1} = p_{i-1} + \Phi_i(p_i) \quad \text{for} \quad i \geq 1.$$

Then, p_i is a coordinate of $K[\mathbf{x}]$ over K for each $i \geq 0$, since $K[p_0, p_1] = K[\mathbf{x}]$ and $R[p_l, p_{l+1}] = R[p_{l-1}, p_l]$ for each $l \geq 1$. We show that p_i satisfies (B) and (C), but is of none of the types I through V when $i \geq 3$. Let c_i be the leading coefficient of $\Phi_i(z)$ for each $i \geq 1$. First, we show that

$$p_i^{\mathbf{w}} = c_1 \cdots c_{i-1} (f_1^{\mathbf{w}})^{e_1 \cdots e_{i-1}} \tag{2.2}$$

holds for each $\mathbf{w} \in \mathbf{N}^2$ and $i \geq 1$ by induction on i. The assertion is clear if $i = 1$. Assume that $i \geq 2$. Then, we have $p_j^{\mathbf{w}} = c_1 \cdots c_{j-1} (f_1^{\mathbf{w}})^{e_1 \cdots e_{j-1}}$ for $j = 1, \ldots, i-1$ by induction assumption. When $i \geq 3$, this implies that $\deg_{\mathbf{w}} p_{i-1} = e_{i-2} \deg_{\mathbf{w}} p_{i-2}$. Since $e_{i-2} \geq 2$, we get $\deg_{\mathbf{w}} p_{i-1} > \deg_{\mathbf{w}} p_{i-2}$. The same holds when $i = 2$ since $\deg_{\mathbf{w}} f_1 > \deg_{\mathbf{w}} x_1$. Hence, $\deg_{\mathbf{w}} \Phi_{i-1}(p_{i-1}) = e_{i-1} \deg_{\mathbf{w}} p_{i-1}$ is greater than $\deg_{\mathbf{w}} p_{i-2}$. Thus, it follows that

$$\begin{aligned} p_i^{\mathbf{w}} &= \bigl(p_{i-2} + \Phi_{i-1}(p_{i-1})\bigr)^{\mathbf{w}} = \Phi_{i-1}(p_{i-1})^{\mathbf{w}} = c_{i-1}(p_{i-1}^{\mathbf{w}})^{e_{i-1}} \\ &= c_{i-1}\bigl(c_1 \cdots c_{i-2} (f_1^{\mathbf{w}})^{e_1 \cdots e_{i-2}}\bigr)^{e_{i-1}} = c_1 \cdots c_{i-1} (f_1^{\mathbf{w}})^{e_1 \cdots e_{i-1}}. \end{aligned}$$

Therefore, (2.2) holds for every $i \geq 1$. Since p_i is a coordinate of $K[\mathbf{x}]$ over K, we may write $p_i^{\mathbf{w}(p_i)}$ as in Proposition 1.2. Thanks to (2.2), this implies that $f_1^{\mathbf{w}(p_i)} = a x_2 + c x_1^\lambda$. Hence, $p_i^{\mathbf{w}(p_i)}$ is a power of $x_2 + (c/a) x_1^\lambda$ multiplied by a nonzero constant. Since f_1 is of type I, we have $\lambda \geq 2$, and c/a does not belong to R. Hence, p_i is tamely reduced over R by Proposition 2.5 (ii). Thus, p_i satisfies (B). By Proposition 1.2, we have $\deg_{x_l} p_i = \deg_{x_l} p_i^{\mathbf{w}(p_i)}$ for $l = 1, 2$. Hence, we see from (2.2) that

$$\deg_{x_1} p_i = \lambda e_1 \cdots e_{i-1} \quad \text{and} \quad \deg_{x_2} p_i = e_1 \cdots e_{i-1}. \tag{2.3}$$

Since $\lambda \geq 2$, it follows that p_i satisfies (C). Moreover, we know that p_i is not of type other than II owing to (1.2). We show that p_i is not of type II. Suppose to the contrary that p_i is of type II. Then, we may find $q \in K[\mathbf{x}]$ such that $\deg_{x_2} q = 1$ and $K[p_i, q] = K[\mathbf{x}]$. Since $K[p_i, p_{i-1}] = K[\mathbf{x}]$, we may write $q = \alpha p_{i-1} + \Psi(p_i)$, where $\alpha \in K^\times$ and $\Psi(z) \in K[z]$. Since $i \geq 3$, we have $\deg_{x_2} p_i > \deg_{x_2} p_{i-1} \geq e_1 \geq 2$ by (2.3). Hence, $\deg_{x_2} q$ is equal to the maximum between $\deg_{x_2} p_{i-1}$ and $\deg_{x_2} \Psi(p_i)$, and is at least $\deg_{x_2} p_{i-1} \geq 2$. This is a contradiction. Thus, p_i is not of type II. Therefore, p_i is of none of the types I through V.

In closing, we construct counterexamples to statements similar to Lemma 1.1 and the "if" part of Theorem 1.2 in the case where R does not contain $\mathbf{Z}$. First, let $R = (\mathbf{Z}/2\mathbf{Z})(z^2)$ and $S = (\mathbf{Z}/2\mathbf{Z})(z)$, and consider the element

$$f := x_2 + x_1^2 + z^2 x_2^2$$

of $R[\mathbf{x}]$. Since $f = x_2 + (x_1 + zx_2)^2$, we can define $\psi \in \mathrm{Aut}(S[\mathbf{x}]/S)$ by $\psi(x_1) = f$ and $\psi(x_2) = x_1 + zx_2$. Hence, f is a coordinate of $S[\mathbf{x}]$ over S. Observe that $f^{\mathbf{w}(f)} = (x_1 + zx_2)^2$. Since z does not belong to $R = K$, this implies that f is not a coordinate of $K[\mathbf{x}]$ over K in view of Proposition 1.2. Therefore, f is a counterexample to a statement similar to Lemma 1.1.

Next, let $R = (\mathbf{Z}/2\mathbf{Z})[z]$, and define $\sigma_i \in \mathrm{Aut}(K[\mathbf{x}]/K)$ for $i = 0, 1, 2$ by $(\sigma_0(x_1), \sigma_0(x_2)) = (x_1, zx_2)$,

$$(\sigma_1(x_1), \sigma_1(x_2)) = (x_1, x_2 + x_1 + x_1^2) \quad \text{and} \quad (\sigma_2(x_1), \sigma_2(x_2)) = (x_1 + x_2^2, x_2).$$

Then, consider the polynomial

$$\begin{aligned} g &:= (\sigma_0 \circ \sigma_1 \circ \sigma_2 \circ \sigma_1)(x_2) \\ &= (\sigma_0 \circ \sigma_1)\big(x_2 + (x_1 + x_2^2) + (x_1^2 + x_2^4)\big) \\ &= \sigma_0\big((x_2 + x_1 + x_1^2) + x_1 + (x_2^2 + x_1^2 + x_1^4) + x_1^2 + (x_2^4 + x_1^4 + x_1^8)\big) \\ &= x_1^2 + x_1^8 + zx_2 + z^2 x_2^2 + z^4 x_2^4. \end{aligned}$$

Clearly, g belongs to $R[\mathbf{x}]$, and is a coordinate of $K[\mathbf{x}]$ over K by definition. We show that g satisfies (A), (B) and (C) of Theorem 1.2, but is of none of the types I through V. Define $\phi \in \mathrm{Aut}(R[\mathbf{x}]/R[x_2])$ by $\phi(x_1) = x_1 + 1$. Then, ϕ belongs to $\mathrm{T}(R, \mathbf{x})$, and satisfies $\phi(g) = g$. Hence, ϕ belongs to $H(g)$. Thus, g satisfies (A). Since $g^{\mathbf{w}(g)} = z^4(x_2 + z^{-1}x_1^2)^4$, and z^{-1} does not belong to R, we know that g is tamely reduced over R by Proposition 2.5 (ii). Hence, g satisfies (B). Since $\deg_{x_1} g = 8$ and $\deg_{x_2} g = 4$, we see that g satisfies (C). Moreover, g is not of type other than II because of (1.2). Since R does not contain a root of unity of positive order, we know that g is not of type II. Therefore, g is of none of the types I through V.

3. Proof (I)

We begin with the proof of the "only if" part of Theorem 1.2. Let $f \in R[\mathbf{x}]$ be a coordinate of $K[\mathbf{x}]$ over K of one of the types I through V. Then, f is tamely reduced over R by the discussion after (1.3). Hence, f satisfies (B). From (1.2), we see that f satisfies (C). By Proposition 1.4, $H(f_i)$ is an infinite group for $i = 1, 3, 4$. Hence, f satisfies (A) if f is of type I or III or IV. We show that H_2 and H_5 are not equal to $\{\mathrm{id}_{R[\mathbf{x}]}\}$. Then,

when Theorem 1.3 is proved, it is also proved that f satisfies (A) if f is of type II or V.

For f_2, set $d = \mathrm{lcm}(e, \mu)$. Then, h belongs to $R'[y_2^d]$, since h belongs to $R'[y_2^e]$ and $R'[y_2^\mu]$ by assumption. By the maximality of μ, it follows that $d = \mu$. Hence, μ is divisible by e. Since $\zeta^e = 1$ by assumption, we get $\zeta^\mu = (\zeta^e)^{\mu/e} = 1$. Because $g_\zeta = (\zeta - 1)(\zeta - 1)^{-1} g = g$ belongs to $R[x_1]$, we know that ζ belongs to Z. Define $\phi \in \mathrm{Aut}(R[\mathbf{x}]/R[x_1])$ by $\phi(x_2) = \zeta x_2 + g$. Then, ϕ belongs to H_2, and is not equal to $\mathrm{id}_{R[\mathbf{x}]}$. Therefore, H_2 is not equal to $\{\mathrm{id}_{R[\mathbf{x}]}\}$. Similarly, since g' belongs to $K[x_1^{e'}]$ and $K[x_1^{\mu'}]$ by assumption, we know that μ' is divisible by e' by the maximality of μ'. Hence, we have $(\zeta')^{\mu'} = ((\zeta')^{e'})^{\mu'/e'} = 1$. Since $\gamma_{i,j}(\zeta')$ belongs to R for $i = 0, 1, 2$ and $j = 1, 2$ by assumption, it follows that ζ' belongs to Z'. Thus, H_5 contains $\phi_{\zeta'}$. Since $\det J\phi_{\zeta'} = \zeta' \neq 1$, we have $\phi_{\zeta'} \neq \mathrm{id}_{R[\mathbf{x}]}$. Therefore, H_5 is not equal to $\{\mathrm{id}_{R[\mathbf{x}]}\}$.

Next, we prove Theorem 1.3. Assume that $i = 1$. Since f_1 is tamely reduced over R, and $\deg_{x_1} f_1 = \lambda$ is greater than $\deg_{x_2} f_1 = 1$, we know that $H(f_1)$ is contained in $J(R; x_1, x_2)$ thanks to Theorem 2.7 (ii). By definition, H_1 is also contained in $J(R; x_1, x_2)$. Hence, it suffices to show that ϕ belongs to $H(f_1)$ if and only if ϕ belongs to H_1 for each $\phi \in J(R; x_1, x_2)$. Take any $\phi \in J(R; x_1, x_2)$. Then, ϕ belongs to $\mathrm{T}(R, \mathbf{x})$. Hence, ϕ belongs to $H(f_1)$ if and only if $\phi(f_1) = f_1$. Since $f_1 = ax_2 + g$ and $a \neq 0$, this condition is equivalent to $a\phi(x_2) + \phi(g) = ax_2 + g$, and to $\phi(x_2) = x_2 + a^{-1}(g - \phi(g))$. This condition is equivalent to the condition that ϕ belongs to H_1. Thus, ϕ belongs to $H(f_1)$ if and only if ϕ belongs to H_1. Therefore, we get $H(f_1) = H_1$.

We treat the case $i = 2$ later. For $i = 3, 4, 5$, we have $\deg_{x_1} f_i = \deg_{x_2} f_i$ by (1.2). Since f_i is tamely reduced over R, we know that $H(f_i)$ is contained in $\mathrm{Aff}(R, \mathbf{x})$ thanks to Theorem 2.7 (i). By definition, H_i is also contained in $\mathrm{Aff}(R, \mathbf{x})$. So we show that ϕ belongs to $H(f_i)$ if and only if ϕ belongs to H_i for each $\phi \in \mathrm{Aff}(R, \mathbf{x})$. Since $\mathrm{Aff}(R, \mathbf{x})$ is contained in $\mathrm{T}(R, \mathbf{x})$, it suffices to check that $\phi(f_i) = f_i$ if and only if ϕ belongs to H_i.

Assume that $i = 3$. Set $f_3' = f_3 - b$. Then, we have $\phi(f_3) = f_3$ if and only if $\phi(f_3') = f_3'$. Write ϕ as in (1.4). Then, since $f_3' = a_1x_1 + a_2x_2 = (x_1, x_2)\mathbf{a}$, we have

$$\begin{aligned}\phi(f_3') &= \phi\big((x_1, x_2)\mathbf{a}\big) = (\phi(x_1), \phi(x_2))\mathbf{a} \\ &= \big((x_1, x_2)A + (b_1, b_2)\big)\mathbf{a} = (x_1, x_2)(A\mathbf{a}) + a_1b_1 + a_2b_2.\end{aligned}$$

Hence, we know that $\phi(f_3') = f_3'$ if and only if

$$(x_1, x_2)(A\mathbf{a}) + a_1b_1 + a_2b_2 = (x_1, x_2)\mathbf{a},$$

and hence if and only if A, b_1 and b_2 satisfy (1.5). Thus, we have $\phi(f_3) = f_3$ if and only if ϕ belongs to H_3. This proves that $H(f_3) = H_3$.

We use the following lemma for the cases of $i = 4, 5$.

Lemma 3.1. *Let $\psi \in \mathrm{Aff}(K, \mathbf{x})$ be such that $\psi(x_2 + g) = x_2 + g$ for some $g \in K[x_1]$ with $\deg_{x_1} g \geq 2$. Then, ψ belongs to $J(K; x_1, x_2)$.*

Proof. It suffices to show that $\deg_{x_2} \psi(x_1) \leq 0$. By assumption, we have $\psi(g) = \psi\big((x_2 + g) - x_2\big) = x_2 + g - \psi(x_2)$. Since $\deg_{x_2} g = 0$ and ψ is

affine, this implies that $\deg_{x_2}\psi(g)\le 1$. On the other hand, we have

$$\deg_{x_2}\psi(g)=(\deg_{x_1}g)\deg_{x_2}\psi(x_1)\ge 2\deg_{x_2}\psi(x_1),$$

since g is a polynomial in the single variable x_1 with $\deg_{x_1}g\ge 2$. Thus, we get $2\deg_{x_2}\psi(x_1)\le 1$. This gives that $\deg_{x_2}\psi(x_1)\le 0$. Therefore, ψ belongs to $J(K;x_1,x_2)$. □

Now, assume that $i=4$. Then, ϕ fixes $f_4=\tau_4(ax_1^2+x_2)$ if and only if $\phi':=\tau_4^{-1}\circ\phi\circ\tau_4$ fixes $ax_1^2+x_2$. Since ϕ' belongs to $\mathrm{Aff}(K,\mathbf{x})$, this implies that ϕ' belongs to $J(K;x_1,x_2)$ by Lemma 3.1, and so implies that $\phi'(x_1)=\epsilon x_1-\alpha$ for some $\epsilon\in K^\times$ and $\alpha\in K$. Hence, ϕ' fixes $ax_1^2+x_2$ if and only if $\phi'(x_1)=\epsilon x_1-\alpha$ and

$$a(\epsilon x_1-\alpha)^2+\phi'(x_2)=\phi'(ax_1^2+x_2)=ax_1^2+x_2 \tag{3.1}$$

for some $\epsilon\in K^\times$ and $\alpha\in K$. Since ϕ' is affine, (3.1) implies that $\epsilon^2=1$. Hence, for $\epsilon\in K^\times$ and $\alpha\in K$, we have (3.1) if and only if ϵ belongs to $\{1,-1\}$ and

$$\phi'(x_2)=2\epsilon\alpha a x_1+x_2-\alpha^2 a.$$

Since $\phi'(x_1)=\epsilon x_1-\alpha$, we may write $(\phi'(x_1),\phi'(x_2))$ as in (1.6). Thus, we know that $\phi(f_4)=f_4$ if and only if (1.6) holds for some $\epsilon\in\{1,-1\}$ and $\alpha\in K$. Therefore, we conclude that $H(f_4)=H_4$.

To prove Theorem 1.3 for $i=5$, we need the following lemma.

Lemma 3.2. *For $\omega\in K^\times$, we have*

$$\phi_\omega(\tau_5(x_1))=\omega\tau_5(x_1),\quad \phi_\omega(\tau_5(x_2))=\tau_5(x_2). \tag{3.2}$$

Proof. Put $\phi=\phi_\omega$, $\tau=\tau_5$ and $\gamma_{i,j}=\gamma_{i,j}(\omega)$ for each i and j. Then, we have

$$\big(\phi(\tau(x_1)),\phi(\tau(x_2))\big)=\big(\phi(x_1),\phi(x_2)\big)\begin{pmatrix}\alpha_1&\beta_1\\ \alpha_2&\beta_2\end{pmatrix}+(\alpha_0,\beta_0) \tag{3.3}$$

and

$$\big(\phi(x_1),\phi(x_2)\big)=(x_1,x_2)\begin{pmatrix}1+\gamma_{1,2}&-\gamma_{1,1}\\ \gamma_{2,2}&1-\gamma_{2,1}\end{pmatrix}+(\gamma_{0,2},-\gamma_{0,1}).$$

A direct computation shows that

$$\begin{aligned}&\begin{pmatrix}1+\gamma_{1,2}&-\gamma_{1,1}\\ \gamma_{2,2}&1-\gamma_{2,1}\end{pmatrix}\begin{pmatrix}\alpha_1&\beta_1\\ \alpha_2&\beta_2\end{pmatrix}\\ &\quad=\begin{pmatrix}\alpha_1+\gamma_{1,2}\alpha_1-\gamma_{1,1}\alpha_2&\beta_1+\gamma_{1,2}\beta_1-\gamma_{1,1}\beta_2\\ \gamma_{2,2}\alpha_1+\alpha_2-\gamma_{2,1}\alpha_2&\gamma_{2,2}\beta_1+\beta_2-\gamma_{2,1}\beta_2\end{pmatrix}=\begin{pmatrix}\omega\alpha_1&\beta_1\\ \omega\alpha_2&\beta_2\end{pmatrix}\end{aligned}$$

and

$$(\gamma_{0,2},-\gamma_{0,1})\begin{pmatrix}\alpha_1&\beta_1\\ \alpha_2&\beta_2\end{pmatrix}=\frac{(\omega-1)\alpha_0}{\alpha_1\beta_2-\alpha_2\beta_1}(\beta_2,-\beta_1)\begin{pmatrix}\alpha_1&\beta_1\\ \alpha_2&\beta_2\end{pmatrix}=\big((\omega-1)\alpha_0,0\big).$$

Hence, the right-hand side of (3.3) is equal to

$$(x_1,x_2)\begin{pmatrix}\omega\alpha_1&\beta_1\\ \omega\alpha_2&\beta_2\end{pmatrix}+\big((\omega-1)\alpha_0,0\big)+(\alpha_0,\beta_0)=\big(\omega\tau(x_1),\tau(x_2)\big).$$

Therefore, we get $\phi(\tau(x_1))=\omega\tau(x_1)$ and $\phi(\tau(x_2))=\tau(x_2)$. □

We show that $\phi(f_5) = f_5$ if ϕ belongs to $H_5 = \{\phi_\omega \mid \omega \in Z'\}$. Take $\omega \in Z'$ such that $\phi = \phi_\omega$. Then, we have $\omega^{\mu'} = 1$. Hence, every element of $K[\tau_5(x_1)^{\mu'}, \tau_5(x_2)]$ is fixed under ϕ_ω by Lemma 3.2. Since g' belongs to $K[x_1^{\mu'}]$, we see that $f_5 = \tau_5(x_2 + g')$ belongs to $K[\tau_5(x_1)^{\mu'}, \tau_5(x_2)]$. Hence, f_5 is fixed under ϕ_ω. Therefore, we get $\phi(f_5) = f_5$.

To prove the converse, we use the following lemma.

Lemma 3.3. *Let f be an element of $K[z] \setminus K$, m the maximal integer for which f belongs to $K[z^m]$, and $\phi \in \mathrm{Aut}(K[z]/K)$ such that $\phi(z) = \alpha z$ for some $\alpha \in K^\times$. If $\phi(f) = f$, then we have $\alpha^m = 1$.*

Proof. Let S be the set of $i \in \mathbf{Z}$ such that the monomial z^i appears in f with nonzero coefficient. Then, we have $\alpha^i = 1$ for each $i \in S$ by the assumption that $\phi(f) = f$. Let m' be the positive generator of the ideal (S) of $\mathbf{Z}$ generated by S. Then, we may write $m' = \sum_{i \in S} i n_i$, where $n_i \in \mathbf{Z}$ for each $i \in S$. Hence, we have $\alpha^{m'} = \prod_{i \in S} (\alpha^i)^{n_i} = 1$. Since (S) is generated by m', we know that S is contained in $m'\mathbf{Z}$. Hence, f belongs to $K[z^{m'}]$. By assumption, f also belongs to $K[z^m]$. Thus, f belongs to $K[z^d]$, where $d := \mathrm{lcm}(m, m')$. By the maximality of m, it follows that $d = m$. Hence, m' divides m. Therefore, we conclude that $\alpha^m = (\alpha^{m'})^{m/m'} = 1$. $\square$

Now, assume that $\phi \in \mathrm{Aff}(R, \mathbf{x})$ fixes $f_5 = \tau_5(x_2 + g')$. Then, $\phi' := \tau_5^{-1} \circ \phi \circ \tau_5$ fixes $x_2 + g'$. Since ϕ' belongs to $\mathrm{Aff}(K, \mathbf{x})$, and $\deg_{x_1} g' \geq 3$ by assumption, it follows that ϕ' belongs to $J(K; x_1, x_2)$ by Lemma 3.1. Write $\phi'(x_1) = \omega x_1 + \beta$, where $\omega \in K^\times$ and $\beta \in K$. We show that $\beta = 0$. Suppose to the contrary that $\beta \neq 0$. Then, the monomial $x_1^{\lambda_2 - 1}$ appears in $\phi'(c_2 x_1^{\lambda_2}) = c_2(\omega x_1 + \beta)^{\lambda_2}$. Set $g'' = g' - c_2 x_1^{\lambda_2}$. Then, we have

$$\phi'(c_2 x_1^{\lambda_2}) = \phi'\big((x_2 + g') - (x_2 + g'')\big) = x_2 + g' - \phi'(x_2 + g'').$$

To obtain a contradiction, it suffices to check that $x_1^{\lambda_2 - 1}$ does not appear in g' and $\phi'(x_2 + g'')$. Since g' belongs to $K[x_1^{\mu'}]$, we know that $\lambda_2 = \deg_{x_1} g'$ is divisible by μ'. Since $\mu' \geq 2$, this implies that $\lambda_2 - 1$ is not divisible by μ'. Hence, $x_1^{\lambda_2 - 1}$ does not appear in g'. Note that $\lambda_2 - 2 \geq 1$ by the assumption that $\lambda_2 \geq 3$. Since g'' is an element of $K[x_1^{\mu'}]$ with $\deg_{x_1} g'' < \lambda_2$, we have $\deg_{x_1} g'' \leq \lambda_2 - \mu' \leq \lambda_2 - 2$. Hence, $x_2 + g''$ has total degree at most $\lambda_2 - 2$. Since an affine automorphism preserves the total degrees, it follows that $\deg \phi'(x_2 + g'') \leq \lambda_2 - 2$. Thus, $x_1^{\lambda_2 - 1}$ does not appear in $\phi'(x_2 + g'')$. Therefore, we are led to a contradiction. This proves that $\beta = 0$. Hence, we have $\phi'(x_1) = \omega x_1$, and so $\phi'(x_1^i) = \omega^i x_1^i$ for each $i \geq 0$. Since g' belongs to $K[x_1^{\mu'}]$, it follows that $g' - \phi'(g')$ belongs to $x_1^{\mu'} K[x_1^{\mu'}]$. On the other hand, we have $g' - \phi'(g') = \phi'(x_2) - x_2$, since $\phi'(x_2 + g') = x_2 + g'$. Hence, $\phi'(x_2) - x_2$ belongs to $x_1^{\mu'} K[x_1^{\mu'}]$. Since ϕ' is affine and $\mu' \geq 2$, this implies that $\phi'(x_2) = x_2$. Consequently, we have $\phi'(g') = g'$. Since μ' is the maximal integer for which g' belongs to $K[x_1^{\mu'}]$, we conclude that $\omega^{\mu'} = 1$ by means of Lemma 3.3. Note that $\phi(\tau_5(x_1)) = \omega \tau_5(x_1)$ and $\phi(\tau_5(x_2)) = \tau_5(x_2)$, since $\phi'(x_1) = \omega x_1$, $\phi'(x_2) = x_2$ and $\phi' = \tau_5^{-1} \circ \phi \circ \tau_5$. By Lemma 3.2, this implies that $\phi = \phi_\omega$. Because ϕ is an element of $\mathrm{Aff}(R, \mathbf{x})$, it follows that $\omega = \det J\phi_\omega$ belongs to $R^\times$. Moreover, $\phi_\omega(x_i) = \phi(x_i)$ belongs to $R[\mathbf{x}]$ for

$i = 1, 2$. From this, we see that $\gamma_{i,j}(\omega)$ belongs to R for $i = 0, 1, 2$ and $j = 1, 2$. Thus, ω belongs to Z'. Therefore, $\phi = \phi_\omega$ belongs to H_5. This completes the proof of Theorem 1.3 for $i = 5$.

Here is a useful identity to be used in the following discussions. Let A be a domain, and γ_i and p_i elements of A for $i = 1, 2$. Define an endomorphism ϕ of the A-algebra $A[z]$ by $\phi(z) = \gamma_1 z + p_1$, and define

$$z' = \gamma_2 z + p_2 \quad \text{and} \quad p' = \gamma_2 p_1 - (\gamma_1 - 1)p_2.$$

Then, it holds that

$$\phi(z') = \gamma_2\phi(z) + p_2 = \gamma_1(\gamma_2 z + p_2) - (\gamma_1 - 1)p_2 + \gamma_2 p_1 = \gamma_1 z' + p'. \tag{3.4}$$

Now, we show that H_2 is contained in $H(f_2)$. Take any $\phi \in H_2$. Then, we have $\phi(x_1) = x_1$ and $\phi(x_2) = \omega x_2 + g_\omega$ for some $\omega \in Z$. In the notation above with $A = R[x_1]$, $z = x_2$ and $(\gamma_1, p_1, \gamma_2, p_2) = (\omega, g_\omega, \zeta - 1, g)$, we have $\phi(z) = \omega x_2 + g_\omega = \phi(x_2)$, $z' = (\zeta - 1)x_2 + g = y_2$ and $p' = (\zeta - 1)g_\omega - (\omega - 1)g = 0$. Hence, we get $\phi(y_2) = \gamma_1 z' + p' = \omega y_2$ by (3.4). Since $\omega^\mu = 1$, it follows that ϕ fixes every element of $K[x_1, y_2^\mu]$. Hence, ϕ fixes $f_2 = ax_1 + h$, since h belongs to $K[y_2^\mu]$ by the choice of μ. Clearly, ϕ belongs to $\mathrm{T}(R, \mathbf{x})$. Thus, ϕ belongs to $H(f_2)$. Therefore, H_2 is contained in $H(f_2)$.

The reverse inclusion is proved by using the following lemma and proposition, where we denote $H(x_1) = \mathrm{Aut}(R[\mathbf{x}]/R[x_1])$ for simplicity.

Lemma 3.4. *$H_2' := H(f_2) \cap H(x_1)$ is contained in H_2.*

PROOF. Take any $\phi \in H_2'$. Then, we have $\phi(x_1) = x_1$. Hence, we may write $\phi(x_2) = \omega x_2 + p$, where $\omega \in R^\times$ and $p \in R[x_1]$. We show that $\omega^\mu = 1$ and $p = g_\omega$. Then, it follows that ϕ belongs to H_2. By applying (3.4) with $A = R[x_1]$, $z = x_2$ and $(\gamma_1, p_1, \gamma_2, p_2) = (\omega, p, \zeta - 1, g)$, we get $\phi(y_2) = \omega y_2 + q$, where

$$q := (\zeta - 1)p - (\omega - 1)g.$$

Then, we have $p = g_\omega$ if and only if $q = 0$. So we prove that $q = 0$. Note that $h = f_2 - ax_1$ is fixed under ϕ, since ϕ belongs to $H(f_2) \cap H(x_1)$. Hence, we have

$$c_1(\omega y_2 + q)^{\lambda_1} = \phi(c_1 y_2^{\lambda_1}) = \phi(h - h') = h - \phi(h'),$$

where $h' := h - c_1 y_2^{\lambda_1}$. Now, suppose to the contrary that $q \neq 0$. Then, the monomial $y_2^{\lambda_1 - 1}$ appears in $(\omega y_2 + q)^{\lambda_1}$ as a polynomial in y_2 over $R'[x_1]$. Hence, $y_2^{\lambda_1 - 1}$ appears in h or $\phi(h')$ by the preceding equality. Since h belongs to $R'[y_2^\mu]$, we know that $\lambda_1 = \deg_{y_2} h$ is divisible by μ. Since $\mu \geq 2$, this implies that $\lambda_1 - 1$ is not divisible by μ. Hence, $y_2^{\lambda_1 - 1}$ does not appear in h. Note that $\deg_{y_2} h' \leq \lambda_1 - \mu \leq \lambda_1 - 2$, since h' is an element of $R'[y_2^\mu]$ with $\deg_{y_2} h' < \lambda_1$. Because $\phi(y_2) = \omega y_2 + q$, it follows that $\deg_{y_2} \phi(h') = \deg_{y_2} h' \leq \lambda_1 - 2$. Hence, $y_2^{\lambda_1 - 1}$ does not appear in $\phi(h')$. This is a contradiction, thus proving $q = 0$. Therefore, we get $p = g_\omega$ and $\phi(y_2) = \omega y_2$. Since $\phi(h) = h$, and μ is the maximal number such that h belongs to $R'[y_2^\mu]$, it follows that $\omega^\mu = 1$ by Lemma 3.3. This proves that ϕ belongs to H_2. Therefore, H_2' is contained in H_2. $\square$

The following proposition is a key to the proof of the "if" part of Theorem 1.2, and Theorem 1.3 for $i = 2$.

Proposition 3.5. *Let $f \in R[\mathbf{x}]$ be a coordinate of $K[\mathbf{x}]$ over K which is tamely reduced over R.*
(i) *Assume that* $\deg_{x_1} f > \deg_{x_2} f \geq 2$. *If* $\phi(f) = f$ *for some* $\phi \in J(R; x_1, x_2)$ *with* $\phi \neq \mathrm{id}_{R[\mathbf{x}]}$, *then* ϕ *belongs to* $H(x_1)$ *and* f *is of type II.*
(ii) *Assume that* $\deg_{x_1} f = \deg_{x_2} f \geq 2$. *If* $\phi(f) = f$ *for some* $\phi \in \mathrm{Aff}(R, \mathbf{x})$ *with* $\phi \neq \mathrm{id}_{R[\mathbf{x}]}$, *then* f *is of type IV, or* f *satisfies the condition* (5) *of Definition 1.1 except for* (c).

We prove Proposition 3.5 in the next section. In the rest of this section, we derive from Proposition 3.5 the "if" part of Theorem 1.2, and Theorem 1.3 for $i = 2$.

Let $f \in R[\mathbf{x}]$ be a coordinate of $K[\mathbf{x}]$ over K which satisfies (A), (B) and (C) of Theorem 1.2. We show that f is of one of the types I through V. Since $\deg_{x_1} f \geq \deg_{x_2} f \geq 1$ by (C), we have $|\mathbf{w}(f)| > 1$. Hence, there exist $l, m \in \mathbf{N}$ and $\alpha, \beta \in K^\times$ such that $\deg_{x_1} f = lm$, $\deg_{x_2} f = m$ and $f^{\mathbf{w}(f)} = \alpha(x_2 + \beta x_1^l)^m$ by Proposition 1.2. Because f is tamely reduced over R by (B), it follows from Proposition 2.5 that β does not belong to $V(R)$ if $l = 1$, and to R if $l \geq 2$.

Assume that $l = m = 1$. Then, we have $\deg_{x_1} f = \deg_{x_2} f = 1$ and $f^{\mathbf{w}(f)} = \alpha(x_2 + \beta x_1)$. Hence, f has the form $\alpha x_2 + \alpha\beta x_1 + b$ for some $b \in R$. Since f belongs to $R[\mathbf{x}]$, it follows that $a_2 := \alpha$ and $a_1 := \alpha\beta$ belong to $R \setminus \{0\}$. Since $l = 1$, we know that $a_1/a_2 = \beta$ does not belong to $V(R)$. Therefore, f is of type III.

Assume that $l \geq 2$ and $m = 1$. Then, we have $\deg_{x_1} f = l$, $\deg_{x_2} f = 1$ and $f^{\mathbf{w}(f)} = \alpha(x_2 + \beta x_1^l)$. From this, we see that $g := f - \alpha x_2$ is an element of $R[x_1]$ of degree $l \geq 2$ with leading coefficient $\alpha\beta$. Since $l \geq 2$, we know that β does not belong to R. Hence, $\alpha\beta$ does not belong to αR. Therefore, f is of type I.

Assume that $l \geq 2$ and $m \geq 2$. Then, we have $\deg_{x_1} f > \deg_{x_2} f \geq 2$. Thanks to Theorem 2.7 (ii), this implies that $H(f)$ is contained in $J(R; x_1, x_2)$ because of (B). By (A), there exists $\phi \in H(f)$ with $\phi \neq \mathrm{id}_{R[\mathbf{x}]}$. Then, ϕ belongs to $J(R; x_1, x_2)$ and satisfies $\phi(f) = f$. Therefore, f is of type II due to Proposition 3.5 (i).

Finally, assume that $l = 1$ and $m \geq 2$. Then, we have $\deg_{x_1} f = \deg_{x_2} f \geq 2$. Thanks to Theorem 2.7 (i), this implies that $H(f)$ is contained in $\mathrm{Aff}(R, \mathbf{x})$ because of (B). By (A), there exists $\phi \in H(f)$ with $\phi \neq \mathrm{id}_{R[\mathbf{x}]}$. Then, ϕ belongs to $\mathrm{Aff}(R, \mathbf{x})$ and satisfies $\phi(f) = f$. Due to Proposition 3.5 (ii), this implies that f is of type IV, or satisfies the condition of Definition 1.1 (5) except for (c).

We show that f is of type V in the latter case. By assumption, we may write $f = \tau_5(x_2 + g')$. Here, τ_5 is an element of $\mathrm{Aff}(K, \mathbf{x})$ satisfying (b) of Definition 1.1 (5), and g' is an element of $x_1^{e'} K[x_1^{e'}]$ for some $e' \geq 2$ with $\deg_{x_1} g' \geq 3$. By Theorem 1.3 for $i = 5$, we have $H(f) = H_5$. Hence, we get $Z' \setminus \{1\} \neq \emptyset$ by (A). Take $\omega \in Z'$ with $\omega \neq 1$. Then, we have $\omega^{\mu'} = 1$ and $\gamma_{i,j}(\omega)$ belongs to R for $i = 0, 1, 2$ and $j = 1, 2$ by definition. Since g' belongs to $K[x_1^{\mu'}]$ and $x_1^{e'} K[x_1^{e'}]$, we know that g' belongs to $x_1^{\mu'} K[x_1^{\mu'}]$. Thus, f satisfies (a) and (c) of Definition 1.1 (5) with e' replaced by μ'. Therefore, f is of type V.

This proves that f is of one of the types I through V. Therefore, the"if" part of Theorem 1.2 follows from Proposition 3.5.

To complete the proof of Theorem 1.3 for $i = 2$, it remains only to prove that $H(f_2)$ is contained in H_2. Thanks to Lemma 3.4, it suffices to verify that $H(f_2)$ is contained in $H(x_1)$. By (1.2), we have $\deg_{x_1} f_2 > \deg_{x_2} f_2 \geq 2$. In view of Theorem 2.7 (ii), this implies that $H(f_2)$ is contained in $J(R; x_1, x_2)$, since f_2 is tamely reduced over R. By Proposition 3.5 (i), we know that

$$\mathrm{Aut}(R[\mathbf{x}]/R[f_2]) \cap J(R; x_1, x_2)$$

is contained in $H(x_1)$. Thus, $H(f_2)$ is contained in $H(x_1)$. This proves that $H(f_2)$ is contained in H_2. Therefore, Proposition 3.5 (i) implies Theorem 1.3 for $i = 2$.

4. Proof (II)

The goal of this section is to prove Proposition 3.5. Recall that elements of $\mathrm{Aut}(R[\mathbf{x}]/R)$ are naturally regarded as elements of $\mathrm{Aut}(A[\mathbf{x}]/A)$ for any R-domain A. For each $\phi \in \mathrm{Aut}(R[\mathbf{x}]/R)$, we define an A-subalgebra of $A[\mathbf{x}]$ by

$$A[\mathbf{x}]^{\phi} = \{f \in A[\mathbf{x}] \mid \phi(f) = f\}.$$

Lemma 4.1. (i) *Let $t \geq 2$ be an integer, and $f \in R[x_1^t, x_2]$ a coordinate of $K[\mathbf{x}]$ over K. Then, we have $f = \alpha x_2 + p$ for some $\alpha \in R \setminus \{0\}$ and $p \in R[x_1^t]$.*
(ii) *Let $\phi \in \mathrm{Aut}(K[\mathbf{x}]/K)$ be such that $\phi(x_1) = \alpha x_1$ and $\phi(x_2) = x_2 + q$ for some $\alpha \in K^{\times}$ and $q \in K[x_1] \setminus \{0\}$ with $\phi(q) = q$. Then, $K[\mathbf{x}]^{\phi}$ is contained in $K[x_1]$.*

Proof. (i) Let U be the set of $f \in K[x_1^t, x_2]$ which is a coordinate of $K[\mathbf{x}]$ over K, but is not of the form $f = \alpha x_2 + p$ for any $\alpha \in K^{\times}$ and $p \in K[x_1^t]$. For each $q \in K[x_1^t]$, we define $\psi_q \in \mathrm{Aut}(K[\mathbf{x}]/K[x_1])$ by $\psi_q(x_2) = x_2 - q$. Then, we remark that $\psi_q(U)$ is contained in U. Now, suppose to the contrary that there exists a coordinate $f' \in R[x_1^t, x_2]$ of $K[\mathbf{x}]$ over K which is not of the form $f' = \alpha x_2 + p$ for any $\alpha \in R \setminus \{0\}$ and $p \in R[x_1^t]$. Then, f' belongs to U. Actually, if $f' = \alpha x_2 + p$ for some $\alpha \in K^{\times}$ and $p \in K[x_1^t]$, then α and p belong to $R \setminus \{0\}$ and $R[x_1^t]$, respectively. Hence, U is not empty. Choose $f \in U$ so that $|\mathbf{w}(f)|$ is minimal. First, we show that $|\mathbf{w}(f)| > 1$. Suppose to the contrary that $|\mathbf{w}(f)| = 1$. Then, f is a linear polynomial in x_i over K for some $i \in \{1, 2\}$. Since f is an element of $K[x_1^t, x_2]$ with $t \geq 2$, we know that $i = 2$. Hence, we have $f = \alpha x_2 + \beta$ for some $\alpha \in K^{\times}$ and $\beta \in K$. Thus, f does not belong to U, a contradiction. Therefore, we get $|\mathbf{w}(f)| > 1$. By Proposition 1.2, there exist $i, j \in \{1, 2\}$ with $i \neq j$, $l, m \in \mathbf{N}$ and $\alpha, \beta \in K^{\times}$ such that $\deg_{x_i} f = m$, $\deg_{x_j} f = lm$ and $f^{\mathbf{w}(f)} = \alpha(x_i + \beta x_j^l)^m$. Since K is of characteristic zero, the monomials $x_i x_j^{l(m-1)}$ and $x_i^{m-1} x_j^l$ appear in $f^{\mathbf{w}(f)}$, and hence appear in f. Because f is an element of $K[x_1^t, x_2]$ with $t \geq 2$, it follows that $(i, j) = (2, 1)$, and l is a multiple of t. Hence, $q := \beta x_1^l$ belongs to $K[x_1^t]$. Put $\psi = \psi_q$. Then, $\psi(U)$ is contained in U as remarked. Hence, $\psi(f)$ belongs to U. Thus, we get $|\mathbf{w}(\psi(f))| \geq |\mathbf{w}(f)|$ by the minimality of $|\mathbf{w}(f)|$. To obtain a contradiction, we show that $|\mathbf{w}(\psi(f))| < |\mathbf{w}(f)|$ using Lemma 2.3. Since f

is a coordinate of $K[\mathbf{x}]$ over K with $|\mathbf{w}(f)| > 0$, we know that f satsfies the equivalent conditions (a) and (b) before Proposition 1.3. By definition, we have $\psi(x_1) = x_1$ and $\psi(x_2) = x_2 - \beta x_1^l$. Since $\mathbf{w}(f) = (m, lm)$, we see that ψ is $\mathbf{w}(f)$-homogeneous. Since

$$\psi(f^{\mathbf{w}(f)}) = \psi\big(\alpha(x_2 + \beta x_1^l)^m\big) = \alpha x_2^m,$$

we have $\deg_{x_1} \psi(f^{\mathbf{w}(f)}) < \deg_{x_1} f^{\mathbf{w}(f)}$. Thus, it follows that $|\mathbf{w}(\psi(f))| < |\mathbf{w}(f)|$ by Lemma 2.3. Therefore, we are led to a contradiction, proving (i).

(ii) Define a K-linear endomorphism of $K[\mathbf{x}]$ by $\delta := \phi - \mathrm{id}_{K[\mathbf{x}]}$. Then, we have $\ker \delta = K[\mathbf{x}]^\phi$, and

$$\delta(x_1^i) = \phi(x_1)^i - x_1^i = \alpha^i x_1^i - x_1^i = (\alpha^i - 1)x_1^i$$

for each $i \in \mathbf{Z}_{\geq 0}$. If $\delta^2(cx_1^i) = c(\alpha^i - 1)^2 x_1^i = 0$ for $c \in K$ and $i \in \mathbf{Z}_{\geq 0}$, then we have $c = 0$ or $\alpha^i = 1$, and hence $\delta(cx_1^i) = c(\alpha^i - 1)x_1^i = 0$. Thus, $\delta^2(f) = 0$ implies $\delta(f) = 0$ for each $f \in K[x_1]$.

Now, take any $p \in K[\mathbf{x}]^\phi \setminus \{0\}$, and write $p = \sum_{i=0}^l p_{l-i} x_2^i$, where $l \in \mathbf{Z}_{\geq 0}$ and $p_0, \ldots, p_l \in K[x_1]$ with $p_0 \neq 0$. Then, it suffices to show that $l = 0$. Since $\phi(K[x_1])$ is contained in $K[x_1]$, and $\phi(x_2) = x_2 + q$, we have

$$\begin{aligned} p = \phi(p) &= \sum_{i=0}^{l} \phi(p_{l-i})(x_2 + q)^i \\ &= \phi(p_0)x_2^l + (l\phi(p_0)q + \phi(p_1))x_2^{l-1} + (\text{terms of lower degree in } x_2). \end{aligned}$$

Hence, we get $\phi(p_0) = p_0$ and $l\phi(p_0)q + \phi(p_1) = p_1$. Thus, p_0 belongs to $K[\mathbf{x}]^\phi$ and

$$\delta(p_1) = \phi(p_1) - p_1 = -l\phi(p_0)q = -lp_0 q.$$

Since q belongs to $K[\mathbf{x}]^\phi$ by assumption, $-lp_0q$ belongs to $K[\mathbf{x}]^\phi = \ker \delta$. Hence, it follows that $\delta^2(p_1) = 0$. This implies that $\delta(p_1) = 0$ as mentioned. Thus, we get $lp_0q = 0$. Since p_0 and q are nonzero, we conclude that $l = 0$ due to the assumption that R contains $\mathbf{Z}$. Therefore, p belongs to $K[x_1]$, proving (ii). □

In the following discussion and two lemmas, $n \in \mathbf{N}$ may be arbitrary. For each endomorphism ϕ of the R-algebra $R[\mathbf{x}]$ and each matrix $A = (a_{i,j})_{i,j}$ with entries in $R[\mathbf{x}]$, we denote $\phi(A) = (\phi(a_{i,j}))_{i,j}$. Then, by chain rule, we have

$$J(\phi \circ \psi) = \phi(J\psi) \cdot J\phi \tag{4.1}$$

for endomorphisms ϕ and ψ of the R-algebra $R[\mathbf{x}]$. If ψ is an automorphism, then we get

$$J(\psi^{-1} \circ \phi \circ \psi) = (\psi^{-1} \circ \phi)(J\psi) \cdot \psi^{-1}(J\phi) \cdot J(\psi^{-1}). \tag{4.2}$$

Now, define an endomorphism ϵ_0 of the R-algebra $R[\mathbf{x}]$ by $\epsilon_0(x_i) = 0$ for $i = 1, \ldots, n$. Assume that $\epsilon_0 \circ \phi = \epsilon_0 \circ \psi = \epsilon_0$, i.e., $\phi(x_i)$ and $\psi(x_i)$ have no constant terms for each i. Then, the matrices $\epsilon_0\big(J(\psi^{-1} \circ \phi \circ \psi)\big)$ and $\epsilon_0(J\phi)$ are similar for the following reason. Since $\epsilon_0 = (\epsilon_0 \circ \psi) \circ \psi^{-1} = \epsilon_0 \circ \psi^{-1}$, we

have

$$\begin{aligned}\epsilon_0(J\psi)\cdot\epsilon_0\big(J(\psi^{-1})\big) &= (\epsilon_0\circ\psi^{-1})(J\psi)\cdot\epsilon_0\big(J(\psi^{-1})\big)\\ &= \epsilon_0\big(\psi^{-1}(J\psi)\cdot J(\psi^{-1})\big) = \epsilon_0\big(J(\psi^{-1}\circ\psi)\big) = E.\end{aligned}$$

Hence, we get $\epsilon_0\big(J(\psi^{-1})\big) = \epsilon_0(J\psi)^{-1}$. Thus, it follows from (4.2) that

$$\begin{aligned}\epsilon_0\big(J(\psi^{-1}\circ\phi\circ\psi)\big) &= \epsilon_0\big((\psi^{-1}\circ\phi)(J\psi)\big)\cdot\epsilon_0\big(\psi^{-1}(J\phi)\big)\cdot\epsilon_0\big(J(\psi^{-1})\big)\\ &= (\epsilon_0\circ\psi^{-1}\circ\phi)(J\psi)\cdot(\epsilon_0\circ\psi^{-1})(J\phi)\cdot\epsilon_0(J\psi)^{-1}\\ &= \epsilon_0(J\psi)\cdot\epsilon_0(J\phi)\cdot\epsilon_0(J\psi)^{-1}.\end{aligned}$$

Therefore, $\epsilon_0(J(\psi^{-1}\circ\phi\circ\psi))$ and $\epsilon_0(J\phi)$ are similar.

Lemma 4.2. *Let κ be any field, and $\phi\in J(\kappa;x_1,\dots,x_n)$ such that $\phi(f)=f$ for some coordinate f of $\kappa[\mathbf{x}]$ over κ. Then, we have $\phi(x_i)=x_i+g$ for some $i\in\{1,\dots,n\}$ and $g\in\kappa[x_1,\dots,x_{i-1}]$.*

Proof. Write $\phi(x_i)=\alpha_i x_i+g_i$ for $i=1,\dots,n$, where $\alpha_i\in\kappa^\times$ and $g_i\in\kappa[x_1,\dots,x_{i-1}]$. We show that $\alpha_i=1$ for some i by contradiction. Suppose that $\alpha_i\neq 1$ for all i. Then, we can define $c_1,\dots,c_n\in\kappa$ by

$$c_i := -(\alpha_i-1)^{-1}g_i(c_1,\dots,c_{i-1})$$

by induction on i. Define $\tau\in\mathrm{Aut}(\kappa[\mathbf{x}]/\kappa)$ by $\tau(x_i)=x_i-c_i$ for $i=1,\dots,n$. Then, we have

$$\begin{aligned}(\phi\circ\tau)(x_i) &= \phi(x_i-c_i) = \alpha_i x_i+g_i-c_i\\ &= \alpha_i(x_i-c_i)+g_i+(\alpha_i-1)c_i = \alpha_i(x_i-c_i)+g_i-g_i(c_1,\dots,c_{i-1})\end{aligned}\tag{4.3}$$

for $i=1,\dots,n$. Set $\phi_0=\tau^{-1}\circ\phi\circ\tau$. Then, we see from (4.3) that $\phi_0(x_i)=\tau^{-1}\big((\phi\circ\tau)(x_i)\big)$ has the form $\alpha_i x_i+g_i'$ for some $g_i'\in\kappa[x_1,\dots,x_{i-1}]$ for each i. Hence, $J\phi_0$ is a lower triangular matrix with diagonal entries $\alpha_1,\dots,\alpha_n$. Thus, the same holds for $\epsilon_0(J\phi_0)$. Therefore, $\alpha_1,\dots,\alpha_n$ are the eigenvalues of $\epsilon_0(J\phi_0)$.

Observe that (4.3) is sent to zero by the substitution $x_j\mapsto c_j$ for $j=1,\dots,i$. Since $(\epsilon_0\circ\tau^{-1})(x_j)=\epsilon_0(x_j+c_j)=c_j$ for $j=1,\dots,n$, it follows that

$$(\epsilon_0\circ\phi_0)(x_i) = (\epsilon_0\circ\tau^{-1})\big((\phi\circ\tau)(x_i)\big) = 0.$$

Hence, we get $\epsilon_0\circ\phi_0=\epsilon_0$. Since f is a coordinate of $\kappa[\mathbf{x}]$ over κ by assumption, there exists $\sigma\in\mathrm{Aut}(\kappa[\mathbf{x}]/\kappa)$ such that $\sigma(x_1)=f$. Define $\sigma_0\in\mathrm{Aut}(\kappa[\mathbf{x}]/\kappa)$ by

$$\sigma_0(x_i) = (\tau^{-1}\circ\sigma)(x_i)-\epsilon_0\big((\tau^{-1}\circ\sigma)(x_i)\big)$$

for $i=1,\dots,n$. Then, we have $\epsilon_0(\sigma_0(x_i))=0$ for each i. Hence, we get $\epsilon_0\circ\sigma_0=\epsilon_0$. Set $\phi_1=\sigma_0^{-1}\circ\phi_0\circ\sigma_0$. Then, $\epsilon_0(J\phi_0)$ and $\epsilon_0(J\phi_1)$ are similar by the discussion above. Accordingly, $\alpha_1,\dots,\alpha_n$ are the eigenvalues of $\epsilon_0(J\phi_1)$. Since $\phi(\sigma(x_1))=\phi(f)=f=\sigma(x_1)$, and $\beta:=\epsilon_0\big((\tau^{-1}\circ\sigma)(x_1)\big)$ is a constant, we have

$$\begin{aligned}(\phi_0\circ\sigma_0)(x_1) &= (\tau^{-1}\circ\phi\circ\tau)\big((\tau^{-1}\circ\sigma)(x_1)-\beta\big)\\ &= \tau^{-1}\big(\phi(\sigma(x_1))\big)-(\tau^{-1}\circ\phi\circ\tau)(\beta) = (\tau^{-1}\circ\sigma)(x_1)-\beta = \sigma_0(x_1).\end{aligned}$$

Hence, we get

$$\phi_1(x_1) = (\sigma_0^{-1} \circ \phi_0 \circ \sigma_0)(x_1) = \sigma_0^{-1}\big((\phi_0 \circ \sigma_0)(x_1)\big) = \sigma_0^{-1}\big(\sigma_0(x_1)\big) = x_1.$$

This shows that the first row of $J\phi_1$ is $(1, 0, \dots, 0)$, and the same holds for $\epsilon_0(J\phi_1)$. Thus, 1 is an eigenvalue of $\epsilon_0(J\phi_1)$. Since $\alpha_1, \dots, \alpha_n$ are the eigenvalues of $\epsilon_0(J\phi_1)$, this contradicts that $\alpha_i \neq 1$ for all i. Therefore, we have $\alpha_i = 1$ for some i. $\square$

We also use locally nilpotent derivations to prove Proposition 3.5. For each domain A, we denote by $Q(A)$ the field of fractions of A. Then, the following facts are well-known. Here, we recall that k is an arbitrary field of characteristic zero. For $D \in \mathrm{LND}_k\, k[\mathbf{x}] \setminus \{0\}$, the transcendence degree of $Q(\ker D)$ over k is equal to $n-1$ (cf. [**6**, Proposition 1.3.32]). If $D(f) \neq 0$ for $D \in \mathrm{Der}_k\, k[\mathbf{x}]$ and $f \in k[\mathbf{x}]$, then f is transcendental over $Q(\ker D)$ (cf. [**18**, Proposition 5.2]).

The following lemma is a consequence of these facts.

Lemma 4.3. *For $D \in \mathrm{LND}_k\, k[\mathbf{x}]$, we put $\phi = \exp D$. Then, we have $k[\mathbf{x}]^\phi = \ker D$.*

PROOF. Obviously, $\ker D$ is contained in $k[\mathbf{x}]^\phi$. We prove the reverse inclusion by contradiction. Suppose that $D(f) \neq 0$ for some $f \in k[\mathbf{x}]^\phi$. Then, f is transcendental over $Q(\ker D)$ as mentioned. Clearly, D is nonzero. Hence, $Q(\ker D)$ has transcendence degree $n-1$ over k as mentioned. Thus, the transcendence degree of $Q(\ker D)(f)$ over k is equal to n. Since $L := Q(k[\mathbf{x}]^\phi)$ contains $Q(\ker D)(f)$, we know that $k(\mathbf{x}) := Q(k[\mathbf{x}])$ is a finite extension of L. Note that ϕ extends to an automorphism of $k(\mathbf{x})$ over L. Hence, ϕ must be of finite order. This contradicts that $\phi = \exp D$ with $D \neq 0$. Thus, $k[\mathbf{x}]^\phi$ is contained in $\ker D$. Therefore, we have $k[\mathbf{x}]^\phi = \ker D$. $\square$

For $\phi \in \mathrm{Aut}(k[\mathbf{x}]/k)$, consider the k-linear endomorphism $\delta = \phi - \mathrm{id}_{k[\mathbf{x}]}$ of $k[\mathbf{x}]$. For each $f, g \in k[\mathbf{x}]$, we have

$$\delta(fg) = \phi(fg) - fg = \big(\phi(f) - f\big)g + \phi(f)\big(\phi(g) - g\big) = \delta(f)g + \phi(f)\delta(g).$$

Since $\delta \circ \phi = \phi \circ \delta$, it follows that

$$\delta^l(fg) = \sum_{i=0}^{l} \binom{l}{i} \phi^i(\delta^{l-i}(f))\delta^i(g)$$

for each $l \geq 1$. By this formula, we know that $\delta^{l+m}(fg) = 0$ if $\delta^l(f) = 0$ and $\delta^m(g) = 0$ for $l, m \in \mathbf{N}$. Hence, we see that δ is locally nilpotent if $\delta^{l_i}(x_i) = 0$ for some $l_i \in \mathbf{N}$ for $i = 1, \dots, n$.

Lemma 4.4. *Let $\phi \in \mathrm{Aut}(k[\mathbf{x}]/k)$ be such that $\phi(x_i) = x_i + g_i$ for some $g_i \in k[x_1, \dots, x_{i-1}]$ for $i = 1, \dots, n$. Then, we have $\phi = \exp D$ for some triangular derivation D of $k[\mathbf{x}]$ over k.*

PROOF. By induction on n, we can check that $\delta = \phi - \mathrm{id}_{k[\mathbf{x}]}$ is locally nilpotent. In fact, assuming that the restriction of δ to $k[x_1, \dots, x_{n-1}]$ is locally nilpotent, we have $\delta^l(x_n) = 0$ for some $l \in \mathbf{N}$, since $\delta(x_n) = (x_n + g_n) - x_n = g_n$ belongs to $k[x_1, \dots, x_{n-1}]$. This implies that δ is locally

nilpotent by the discussion above. Due to van den Essen [**6**, Proposition 2.1.3] and Nowicki [**23**, Proposition 6.1.4 (6)], it follows that

$$D := \sum_{i\geq 1}(-1)^{i+1}\frac{\delta^i}{i}$$

belongs to $\mathrm{LND}_k\, k[\mathbf{x}]$ and satisfies $\phi = \exp D$. Since $\delta(x_i) = (x_i + g_i) - x_i = g_i$ belongs to $k[x_1, \ldots, x_{i-1}]$, and $\delta(k[x_1, \ldots, x_{i-1}])$ is contained in $k[x_1, \ldots, x_{i-1}]$, we see that $D(x_i)$ belongs to $k[x_1, \ldots, x_{i-1}]$ for $i = 1, \ldots, n$. Therefore, D is triangular. □

Now, we prove Proposition 3.5 (i). Let $f \in R[\mathbf{x}]$ be a coordinate of $K[\mathbf{x}]$ over K which is tamely reduced over R and satisfies $\deg_{x_1} f > \deg_{x_2} f \geq 2$. Take $\mathrm{id}_{R[\mathbf{x}]} \neq \phi \in J(R; x_1, x_2)$ such that $\phi(f) = f$, and write

$$\phi(x_1) = u_1x_1 + v \quad\text{and}\quad \phi(x_2) = u_2x_2 + g, \tag{4.4}$$

where $u_1, u_2 \in R^\times$, $v \in R$ and $g \in R[x_1]$. Then, we have $u_1 = 1$ or $u_2 = 1$ by Lemma 4.2. Since f is a coordinate of $K[\mathbf{x}]$ over K with $\deg_{x_1} f > \deg_{x_2} f \geq 2$, we see from Proposition 1.2 that f has the form $\alpha x_2^l +$ (terms of lower degree in x_2) for some $\alpha \in R \setminus \{0\}$ and $l \geq 2$. Then, we have

$$\phi(f) = \alpha u_2^l x_2^l + (\text{terms of lower degree in } x_2)$$

because of (4.4). Since $\phi(f) = f$, it follows that $u_2^l = 1$. Hence, u_2 is a root of unity. We show that $u_1 = 1$ and $u_2 \neq 1$ by contradiction.

Suppose that $u_1 = u_2 = 1$. Then, there exists a triangular derivation D of $K[\mathbf{x}]$ over K such that $\phi = \exp D$ by Lemma 4.4. Here, we identify ϕ with the natural extension to $K[\mathbf{x}]$. Since $\phi \neq \mathrm{id}_{R[\mathbf{x}]}$, we have $D \neq 0$. Assume that $D(x_1) = 0$. Then, we have $D(x_2) \neq 0$. Hence, we get $\ker D = K[x_1]$. Since f belongs to $K[\mathbf{x}]^\phi$, and $K[\mathbf{x}]^\phi = \ker D$ by Lemma 4.3, it follows that f belongs to $K[x_1]$. Hence, f is a linear polynomial in x_1 over K. This contradicts that $\deg_{x_2} f \geq 2$. If $D(x_1) \neq 0$, then $D(x_1)$ belongs to $K^\times$ by the triangularity of D. By integrating $D(x_1)^{-1}D(x_2)$ in x_1, we can construct $q \in K[x_1]$ such that $dq/dx_1 = D(x_1)^{-1}D(x_2)$. Then, we have $D(x_2 - q) = D(x_2) - D(x_1)dq/dx_1 = 0$. Since $x_2 - q$ is a coordinate of $K[\mathbf{x}]$ over K, it follows that $\ker D = K[x_2 - q]$ by Theorem 2.1. Hence, f belongs to $K[x_2 - q]$. Since f is a coordinate, f must be a linear polynomial in $x_2 - q$ over K. Thus, we get $\deg_{x_2} f = 1$, a contradiction. Therefore, we have $(u_1, u_2) \neq (1, 1)$.

Next, suppose that $u_1 \neq 1$ and $u_2 = 1$. Set

$$z_1 = (u_1 - 1)x_1 + v.$$

Then, we have $K[z_1] = K[x_1]$ since $u_1 \neq 1$. Moreover, we get

$$\phi(z_1) = u_1z_1 + (u_1 - 1)v - (u_1 - 1)v = u_1z_1$$

by applying (3.4) with $(\gamma_1, p_1, \gamma_2, p_2) = (u_1, v, u_1 - 1, v)$. Since g belongs to $K[x_1] = K[z_1]$, we may write $g = \sum_{i\geq 0}\beta_i z_1^i$, where $\beta_i \in K$ for each i. Let I be the set of i such that $\beta_i \neq 0$ and $u_1^i \neq 1$. Then, define

$$z_2 = x_2 + q, \quad \text{where} \quad q := \sum_{i\in I}\beta_i(1 - u_1^i)^{-1}z_1^i.$$

Since $u_2 = 1$, we have

$$\phi(z_2) = \phi(x_2 + q) = (x_2 + g) + \phi(q) = (z_2 - q) + g + \phi(q) = z_2 + q',$$

where

$$\begin{aligned} q' := g + \phi(q) - q &= \sum_{i\geq 0} \beta_i z_1^i + \sum_{i\in I} \beta_i (1-u_1^i)^{-1}(\phi(z_1^i) - z_1^i) \\ &= \sum_{i\geq 0} \beta_i z_1^i + \sum_{i\in I} \beta_i (1-u_1^i)^{-1}(u_1^i - 1) z_1^i = \sum_{i\notin I} \beta_i z_1^i. \end{aligned}$$

Note that i does not belong to I if and only if $\beta_i = 0$ or $u_1^i = 1$, and so if and only if $\phi(\beta_i z_1^i) = \beta_i u_1^i z_1^i$ is equal to $\beta_i z_1^i$. Hence, we get $\phi(q') = q'$. If $q' \neq 0$, then we know by Lemma 4.1 (ii) that $K[\mathbf{x}]^\phi$ is contained in $K[z_1] = K[x_1]$. Hence, f is a linear polynomial in x_1 over K, a contradiction. If $q' = 0$, then we have $\phi(z_2) = z_2$. Since $\phi(z_1) = u_1 z_1$ with $u_1 \neq 1$, it follows that $K[\mathbf{x}]^\phi = K[z_1, z_2]^\phi$ is equal to $K[z_1^t, z_2]$ for some $t \geq 2$ if u_1 is a root of unity, and to $K[z_2]$ otherwise. In the former case, f is a linear polynomial in z_2 over $K[z_1^t]$ by Lemma 4.1 (i). In the latter case, f is a linear polynomial in z_2 over K. In either case, we have $\deg_{x_2} f = 1$, a contradiction. Therefore, we conclude that $u_1 = 1$ and $u_2 \neq 1$.

Since u_2 is a root of unity as mentioned, we may find the maximal integer $e \geq 2$ such that $u_2^e = 1$. Then, we have

$$\phi^e(x_1) = x_1 + ev \quad \text{and} \quad \phi^e(x_2) = x_2 + p$$

for some $p \in R[x_1]$. Hence, ϕ^e belongs to $J(R; x_1, x_2)$. Since $\phi^e(f) = f$, we may conclude that $\phi^e = \mathrm{id}_{R[\mathbf{x}]}$ from the discussion for the case of $u_1 = u_2 = 1$. Hence, we have $ev = 0$, and so $v = 0$. Thus, we get $\phi(x_1) = x_1$. Therefore, ϕ belongs to $H(x_1)$, proving the first part of Proposition 3.5 (i).

We check that f is of type II. Set $u = u_2 - 1$, and define

$$y_2 = ux_2 + g \quad \text{and} \quad R' = R[u^{-1}].$$

Then, we have $R'[x_1, y_2] = R'[\mathbf{x}]$ since $u_2 \neq 1$. Moreover, we get $\phi(y_2) = u_2 y_2$ by applying (3.4) with $(\gamma_1, p_1, \gamma_2, p_2) = (u_2, g, u, g)$. Hence, it follows that

$$R'[\mathbf{x}]^\phi = R'[x_1, y_2]^\phi = R'[x_1, y_2^e]$$

by the definition of e. Since f belongs to this set, we know by Lemma 4.1 (i) that $f = a'x_1 + h$ for some $a' \in R' \setminus \{0\}$ and $h \in R'[y_2^e]$. Then, h does not belong to R' by the assumption that $\deg_{x_2} f \geq 2$. Furthermore, we have $\lambda := \deg_{x_1} g \geq 2$ by the assumption that $\deg_{x_1} f > \deg_{x_2} f$. Let c be the leading coefficient of g. Then, $f^{\mathbf{w}(f)}$ is equal to a power of $y_2^{\mathbf{w}(y_2)} = ux_2 + cx_1^\lambda$ multiplied by an element of $R' \setminus \{0\}$. Since $\lambda \geq 2$ and f is tamely reduced over R by assumption, it follows from Proposition 2.5 (ii) that c does not belong to uR. Thus, f satisfies all the conditions of Definition 1.1 (2). Therefore, f is of type II. This completes the proof of Proposition 3.5 (i).

We use the following lemma to prove Proposition 3.5 (ii).

Lemma 4.5. *Let $f \in R[\mathbf{x}]$ be such that $f = \psi(\gamma x_2 + q)$ for some $\psi \in \mathrm{Aff}(K, \mathbf{x})$, $\gamma \in K^\times$ and $q \in K[x_1]$. If f is tamely reduced over R, and $\deg_{x_1} f = \deg_{x_2} f$, then the following statements hold*:

(i) *If $\deg_{x_1} q = 2$, then f is of type IV.*

(ii) *Assume that q belongs to $K[(sx_1+t)^l]\setminus K$ for some $s\in K^\times$, $t\in K$ and $l\geq 2$. If $\deg_{x_1} q\geq 3$, then f satisfies the conditions of Definition 1.1* (5) *except for* (c).

Proof. Write $\psi(x_1)=\alpha_1'x_1+\alpha_2'x_2+\alpha_0'$, where $\alpha_1',\alpha_2',\alpha_0'\in K$. If $\deg_{x_1} q=2$, then we may write

$$q=\beta_2(x_1-\alpha_0')^2+\beta_1x_1+\beta_0,$$

where $\beta_0,\beta_1,\beta_2\in K$ with $\beta_2\neq 0$. Take $u\in K^\times$ such that $a:=u^2\beta_2$ belongs to R. Then, we can define $\tau\in \mathrm{Aff}(K,\mathbf{x})$ by

$$\tau(x_1)=u^{-1}\psi(x_1-\alpha_0'),\quad \tau(x_2)=\psi(\beta_1x_1+\gamma x_2+\beta_0),$$

since $\gamma\neq 0$. Using this τ, we may write

$$f=\psi(q+\gamma x_2)=\psi\big(\beta_2(x_1-\alpha_0')^2+\beta_1x_1+\beta_0+\gamma x_2\big)=a\tau(x_1)^2+\tau(x_2).$$

Put $\alpha_i=u^{-1}\alpha_i'$ for $i=1,2$. Then, we have $\tau(x_1)=\alpha_1x_1+\alpha_2x_2$. Since $\deg_{x_1} f=\deg_{x_2} f$ by assumption, we know that α_1 and α_2 are nonzero. Hence, we get

$$f^{\mathbf{w}(f)}=a\tau(x_1)^2=a\alpha_2^2\big((\alpha_1/\alpha_2)x_1+x_2\big)^2.$$

Since f is tamely reduced over R by assumption, this implies that α_1/α_2 does not belong to $V(R)$ by Proposition 2.5 (i). Therefore, f is of type IV.

Next, assume that q is as in (ii). Put $y_1=sx_1+t$, and write $q=q'+\delta$, where $q'\in y_1^lK[y_1^l]$ and $\delta\in K$. Let g' be an element of $x_1^lK[x_1^l]$ obtained from q' by replacing y_1 with x_1. Then, we have $\lambda:=\deg_{x_1} g'=\deg_{y_1} q'=\deg_{y_1} q\geq 3$. Define $\tau\in\mathrm{Aff}(K,\mathbf{x})$ by $\tau(x_1)=\psi(y_1)$ and $\tau(x_2)=\psi(\gamma x_2+\delta)$. Then, we have

$$f=\psi(\gamma x_2+q'+\delta)=\psi(\gamma x_2+\delta)+\tau(g')=\tau(x_2+g'). \tag{4.5}$$

Hence, f is written as in Definition 1.1 (5), and satisfies (a). Put $\alpha_i:=s\alpha_i'$ for $i=1,2$. Then, we have

$$\tau(x_1)=\psi(sx_1+t)=\alpha_1x_1+\alpha_2x_2+s\alpha_0'+t.$$

Since $\deg_{x_1} f=\deg_{x_2} f$ by assumption, we see from (4.5) that α_1 and α_2 are nonzero, and

$$f^{\mathbf{w}(f)}=(g')^{\mathbf{w}(f)}=c(\alpha_1x_1+\alpha_2x_2)^\lambda=c\alpha_2^\lambda((\alpha_1/\alpha_2)x_1+x_2)^\lambda,$$

where $c\in K^\times$ is the leading coefficient of g'. Since f is tamely reduced over R by assumption, this implies that α_1/α_2 does not belong to $V(R)$ by Proposition 2.5 (ii). Thus, (b) of Definition 1.1 (5) is satisfied. Therefore, f satisfies the conditions of Definition 1.1 (5) except for (c). $\square$

Now, we prove Proposition 3.5 (ii). Let $f\in R[\mathbf{x}]$ be a coordinate of $K[\mathbf{x}]$ over K which is tamely reduced over R and satisfies $\deg_{x_1} f=\deg_{x_2} f\geq 2$. Assume that $\phi(f)=f$ for some $\mathrm{id}_{R[\mathbf{x}]}\neq\phi\in\mathrm{Aff}(R,\mathbf{x})$. Then, it suffices to show that f is written as in Lemma 4.5. Write $(\phi(x_1),\phi(x_2))=(x_1,x_2)A+(b_1,b_2)$, where $A\in GL(2,R)$ and $b_1,b_2\in R$. Let K' be an extension field of K to which the eigenvalues of A belong. Then, we know from linear algebra that $P^{-1}AP$ is upper triangular for some $P\in GL(2,K')$. We define

$\psi \in \operatorname{Aff}(K', \mathbf{x})$ by $(\psi(x_1), \psi(x_2)) = (x_1, x_2)P$, and put $\phi' = \psi^{-1} \circ \phi \circ \psi$. Then, we have

$$\begin{aligned}(\phi'(x_1), \phi'(x_2)) &= \big((\psi^{-1} \circ \phi)(x_1), (\psi^{-1} \circ \phi)(x_2)\big)P \\ &= \big((\psi^{-1}(x_1), \psi^{-1}(x_2))A + (b_1, b_2)\big)P = (x_1, x_2)P^{-1}AP + (b_1, b_2)P.\end{aligned}$$

Since $P^{-1}AP$ is regular and upper triangular, we may write

$$\phi'(x_1) = u_1x_1 + \beta_1 \quad \text{and} \quad \phi'(x_2) = vx_1 + u_2x_2 + \beta_2,$$

where $u_1, u_2 \in (K')^\times$ are the eigenvalues of A, and $v, \beta_1, \beta_2 \in K'$. Hence, ϕ' belongs to $J(K'; x_1, x_2)$. Since $\phi(f) = f$ by assumption, ϕ' fixes the coordinate $\psi^{-1}(f)$ of $K'[\mathbf{x}]$ over K'. Thus, we have $u_1 = 1$ or $u_2 = 1$ by Lemma 4.2. Since $u_1u_2 = \det P^{-1}AP = \det A$ belongs to $R^\times$, it follows that u_1 and u_2 belong to $R^\times$. Hence, we may assume that $K' = K$. Since $u_1 = 1$ or $u_2 = 1$, we have $u_1 \neq u_2$ if and only if $(u_1, u_2) \neq (1, 1)$. If this is the case, then we may choose P so that $P^{-1}AP$ is a diagonal matrix. Then, we have $v = 0$. In this case, we may assume further that $u_1 \neq 1$ and $u_2 = 1$ by replacing P if necessary. Thus, we are reduced to the following two cases:

$$\begin{cases} \phi'(x_1) & = x_1 + \beta_1 \\ \phi'(x_2) & = x_2 + \alpha x_1 + \beta_2, \end{cases} \qquad \begin{cases} \phi'(x_1) & = ux_1 + \beta_1 \\ \phi'(x_2) & = x_2 + \beta_2. \end{cases}$$

Here, α, β_1 and β_2 are elements of K, and $u \neq 1$ is an element of $R^\times$.

First, we consider the former case. Define $D \in \operatorname{Der}_K K[\mathbf{x}]$ by

$$D(x_1) = \beta_1 \quad \text{and} \quad D(x_2) = \alpha x_1 + \beta_2 - \frac{\alpha\beta_1}{2}.$$

Then, D is triangular, and satisfies $\exp D = \phi'$, since

$$\begin{aligned}&(\exp D)(x_1) = x_1 + D(x_1) = x_1 + \beta_1 \\ &(\exp D)(x_2) = x_2 + D(x_2) + \frac{D^2(x_2)}{2} = x_2 + \left(\alpha x_1 + \beta_2 - \frac{\alpha\beta_1}{2}\right) + \frac{\alpha\beta_1}{2}.\end{aligned}$$

Hence, we have $K[\mathbf{x}]^{\phi'} = \ker D$ by Lemma 4.3, and so $K[\mathbf{x}]^{\phi} = \psi(\ker D)$. Since f is a coordinate of $K[\mathbf{x}]$ over K belonging to $K[\mathbf{x}]^\phi$, we know that, if $\ker D = K[h]$ for some $h \in K[\mathbf{x}]$, then f is a linear polynomial in $\psi(h)$ over K.

We prove that $\beta_1 \neq 0$ by contradiction. If $\beta_1 = 0$, then we have $\ker D = K[x_1]$. Hence, f is a linear polynomial in $\psi(x_1)$ over K. Since ψ is affine, it follows that $\deg f = 1$, a contradiction. Thus, we get $\beta_1 \neq 0$. Define

$$q = \frac{\alpha}{2\beta_1}x_1^2 + \left(\frac{\beta_2}{\beta_1} - \frac{\alpha}{2}\right)x_1.$$

Then, we have $D(x_2 - q) = 0$, since

$$D(q) = D(x_1)\frac{\partial q}{\partial x_1} = \beta_1\left(2\frac{\alpha}{2\beta_1}x_1 + \frac{\beta_2}{\beta_1} - \frac{\alpha}{2}\right) = D(x_2).$$

Since $x_2 - q$ is a coordinate of $K[\mathbf{x}]$ over K, it follows that $\ker D = K[x_2 - q]$ by Theorem 2.1. Hence, we have

$$f = \gamma\psi(x_2 - q) + \gamma' = \psi\big(\gamma x_2 + (\gamma' - \gamma q)\big)$$

for some $\gamma \in K^\times$ and $\gamma' \in K$ as remarked. Since $\deg f \geq 2$ by assumption, this implies that $\deg_{x_1} q \geq 2$. Thus, we get $\alpha \neq 0$, and so $\deg_{x_1}(\gamma' - \gamma q) = \deg_{x_1} q = 2$. Therefore, f is expressed as in Lemma 4.5 (i).

Next, we consider the latter case. Set $y_1 = (u-1)x_1 + \beta_1$. Then, we have $K[\mathbf{x}] = K[y_1, x_2]$ since $u \neq 1$. Moreover, we get $\phi'(y_1) = uy_1$ by applying (3.4) with $(\gamma_1, p_1, \gamma_2, p_2) = (u, \beta_1, u-1, \beta_1)$. We prove that $\beta_2 = 0$ and u is a root of unity. Suppose to the contrary that $\beta_2 \neq 0$. Then, $K[\mathbf{x}]^{\phi'} = K[y_1, x_2]^{\phi'}$ is contained in $K[y_1]$ by Lemma 4.1 (ii). Hence, f belongs to $\psi(K[y_1])$. This implies that f is a linear polynomial in $\psi(y_1)$ over K. Since ψ is affine, it follows that $\deg f = \deg \psi(y_1) = \deg y_1 = 1$, a contradiction. Thus, we get $\beta_2 = 0$, and so $\phi'(x_2) = x_2$. Suppose that u is not a root of unity. Then, we have $K[\mathbf{x}]^{\phi'} = K[y_1, x_2]^{\phi'} = K[x_2]$, since $\phi'(y_1) = uy_1$. Hence, f belongs to $\psi(K[x_2])$. This implies that f is a linear polynomial in $\psi(x_2)$ over K. Thus, we get $\deg f = 1$, a contradiction. Therefore, u is a root of unity. Consequently, we have $K[\mathbf{x}]^{\phi'} = K[y_1^l, x_2]$ for some $l \geq 2$. Note that $\psi^{-1}(f)$ belongs to $K[\mathbf{x}]^{\phi'}$, and is a coordinate of $K[\mathbf{x}]$ over K. Hence, by virtue of Lemma 4.1 (i), we may write $\psi^{-1}(f) = \gamma x_2 + q$, where $\gamma \in K^\times$ and $q \in K[y_1^l]$. Then, we have $f = \psi(\gamma x_2 + q)$. Since $\deg f \geq 2$ by assumption, we see that q does not belong to K. Hence, we know that $\deg_{x_1} q = \deg_{y_1} q \geq l \geq 2$. If $\deg_{x_1} q = 2$, then f is expressed as in Lemma 4.5 (i). If $\deg_{x_1} q \geq 3$, then f is expressed as in Lemma 4.5 (ii). This completes the proof of Proposition 3.5 (ii), and thereby completing the proof of Theorems 1.2 and 1.3.

5. Application

Assume that R is a $\mathbf{Q}$-domain, and let D be a triangular derivation of $R[\mathbf{x}]$ over R with $D(x_i) \neq 0$ for $i = 1, 2$. Take any $f \in \ker D \setminus R$, and put $\phi = \exp fD$. In this section, we investigate when the coordinates $\phi(x_1)$ and $\phi(x_2)$ of $R[\mathbf{x}]$ over R are totally wild or quasi-totally wild. Note that $\phi(x_i)$ is quasi-totally wild if and only if $\phi(x_i)$ is exponentially wild for $i = 1, 2$ by Corollary 1.5 (i). If $\phi(x_i)$ is exponentially wild, then $\phi(x_i)$ is wild as mentioned after Definition 0.1. If $\phi(x_i)$ is wild for some $i \in \{1, 2\}$, then ϕ does not belong to $\mathrm{T}(R, \mathbf{x})$ by definition. Conversely, if ϕ does not belong to $\mathrm{T}(R, \mathbf{x})$, then $\phi(x_i)$ is wild for $i = 1, 2$. In fact, if $\phi(x_i)$ is tame for some $i \in \{1, 2\}$, and $\tau \in \mathrm{T}(R, \mathbf{x})$ is such that $\tau(x_i) = \phi(x_i)$, then $\tau^{-1} \circ \phi$ belongs to $\mathrm{Aut}(R[\mathbf{x}]/R[x_i])$.

Write D as in (2.1), and define I as in (2.3). Then, due to Theorem 2.2, ϕ does not belong to $\mathrm{T}(R, \mathbf{x})$ if and only if one of the following conditions holds:

(W1) $I \cap \{1, \ldots, l\} \neq \emptyset$.

(W2) $I = \{0\}$, b_0/a does not belong to $V(R)$ and $\deg_{x_2} f \neq 1$.

Here, we note that $\deg_{x_2} f \neq 1$ if and only if $\deg_{x_2} f \geq 2$, since f is not an element of R by assumption.

Define $\tau \in \mathrm{Aut}(R[\mathbf{x}]/R[x_1])$ as in (2.4), and set $g_i = \tau(\phi(x_i))$ for $i = 1, 2$. Then, $\phi(x_i)$ is totally (resp. quasi-totally) wild if and only if g_i is totally (resp. quasi-totally) wild for $i = 1, 2$.

With this notation, we have the following theorems.

Theorem 5.1. *Assume that* $\phi = \exp fD$ *does not belong to* $\mathrm{T}(R, \mathbf{x})$. *Then, the following assertions hold*:
(i) $\deg_{x_1} g_1 \geq 2$, $\deg_{x_1} g_1 \geq \deg_{x_2} g_2 \geq 1$, *and* $\deg_{x_1} g_2 \geq \deg_{x_2} g_2 \geq 2$.
(ii) g_1 *and* g_2 *are tamely reduced over* R.
(iii) *Assume that* $H(g_i)$ *is not equal to* $\{\mathrm{id}_{R[\mathbf{x}]}\}$ *for some* $i \in \{1, 2\}$. *Then,* g_i *is of type I or IV or V if* $i = 1$, *and of type IV or V if* $i = 2$.

Next, consider the following condition:

$$I = \{0\} \text{ and } b_0/a \text{ does not belong to } V(R), \text{ and } l = 0 \text{ if } i = 2. \tag{5.1}$$

Theorem 5.2. *The following assertions hold*:
(i) g_1 *is of type I if and only if* $\deg_{x_2} f = 1$ *and* $I \cap \{1, \ldots, l\} \neq \emptyset$.
(ii) g_i *is of type IV for* $i \in \{1, 2\}$ *if and only if* $\deg_{x_2} f = 2$ *and (5.1) holds.*
(iii) *If* g_i *is of type V for* $i \in \{1, 2\}$, *then we have* $\deg_{x_2} f \geq 3$ *and (5.1).*

First, we describe $\phi(x_1)$ and $\phi(x_2)$ concretely. Since $D(x_1) = a$, we have

$$\phi(x_1) = (\exp fD)(x_1) = x_1 + fD(x_1) = x_1 + af.$$

Define h as in (2.2). Then, we have $\phi(h) = h$, since $D(h) = 0$. Hence, we get

$$ax_2 - \sum_{i=0}^{l} \frac{b_i}{i+1} x_1^{i+1} = h = \phi(h) = a\phi(x_2) - \sum_{i=0}^{l} \frac{b_i}{i+1}(x_1 + af)^{i+1}.$$

This gives that

$$\phi(x_2) = x_2 + \sum_{i=0}^{l} \frac{b_i}{(i+1)a}\big((x_1 + af)^{i+1} - x_1^{i+1}\big).$$

Set $f_0 = \tau(f)$. Then, we have

$$\begin{aligned} g_1 &= \tau(\phi(x_1)) = x_1 + af_0 \\ g_2 &= \tau(\phi(x_2)) = x_2 + \sum_{i=0}^{l} \frac{b_i}{(i+1)a}(x_1 + af_0)^{i+1} - \sum_{i \in I} \frac{b_i}{(i+1)a} x_1^{i+1}, \end{aligned} \tag{5.2}$$

since I is the complement of I' in $\{0, 1, \ldots, l\}$.

Now, we prove Theorems 5.1 and 5.2. Since ϕ does not belong to $\mathrm{T}(R, \mathbf{x})$ by assumption, we have $I \neq \emptyset$ by Theorem 2.2. Set $t = \max I \geq 0$ and $h_0 = \tau(h)$. Then, we see from (2.5) that $h_0^{\mathbf{w}(h_0)}$ is written as in (2.7). Since $\ker D$ is contained in $K[h]$ as mentioned before Theorem 2.1, f_0 belongs to $K[h_0] \setminus R$. Hence, we have $m := \deg_{x_2} f = \deg_h f = \deg_{h_0} f_0 \geq 1$. We show that

$$g_1^{\mathbf{w}(g_1)} = c_1(x_2 - bx_1^{t+1})^m, \quad g_2^{\mathbf{w}(g_2)} = c_2(x_2 - bx_1^{t+1})^{(l+1)m} \tag{5.3}$$

for some $c_1, c_2 \in K^\times$, where $b = ((t+1)a)^{-1} b_t$. Since $f_0^{\mathbf{w}(f_0)}$ is equal to $(h_0^{\mathbf{w}(h_0)})^m = a^m(x_2 - bx_1^{t+1})^m$ up to a nonzero constant multiple, it suffices to verify that $\deg_{x_1} f_0 \geq 2$, $\deg_{x_2} f_0 \geq 1$ and $\deg_{x_2} f_0^{l+1} \geq 2$ in view of (5.2). Note that $\deg_{x_1} h_0 = t + 1$, $\deg_{x_2} h_0 = 1$ and

$$\deg_{x_i} f_0 = (\deg_{h_0} f_0) \deg_{x_i} h_0 = m \deg_{x_i} h_0$$

for $i = 1, 2$. First, assume that $t \geq 1$. Then, it follows that $\deg_{x_1} f_0 \geq 2m \geq 2$ and $\deg_{x_2} f_0 = m \geq 1$. Since I is a subset of $\{0, \ldots, l\}$, we have

$l \geq t$. Hence, we get $\deg_{x_2} f_0^{l+1} = (l+1)m \geq 2$. Thus, the assertion holds. Next, assume that $t = 0$. Then, we have $I = \{0\}$. Since ϕ does not belong to $\mathrm{T}(R, \mathbf{x})$ by assumption, this implies that ϕ satisfies (W2). Hence, we get $m = \deg_{x_2} f \geq 2$. Since $\deg_{x_i} h_0 = 1$ for $i = 1, 2$, it follows that $\deg_{x_i} f_0 = m \geq 2$ for $i = 1, 2$, and $\deg_{x_2} f_0^{l+1} = (l+1)m \geq m \geq 2$. Thus, the assertion holds. Therefore, we obtain (5.3).

Thanks to Proposition 1.2, we see from (5.3) that

$$\begin{array}{ll} \deg_{x_1} g_1 = (t+1)m & \deg_{x_1} g_2 = (t+1)(l+1)m \\ \deg_{x_2} g_1 = m & \deg_{x_2} g_2 = (l+1)m. \end{array} \tag{5.4}$$

By the preceding discussion, we have $l \geq t$, and $m \geq 2$ if $t = 0$. Hence, (i) follows from (5.4). We prove (ii) using Proposition 2.5. First, assume that $t = 0$. Then, we have $b = b_0/a$. Since $I = \{0\}$, we know that b does not belong to $V(R)$ by (W2). By (5.3), this implies that g_1 and g_2 are tamely reduced over R because of Proposition 2.5 (i). Next, assume that $t \geq 1$. Since t is an element of I, we see that b_t does not belong to aR. Hence, $b = ((t+1)a)^{-1} b_t$ does not belong to R. By (5.3), this implies that g_1 and g_2 are tamely reduced over R because of Proposition 2.5 (ii). This proves (ii).

To prove (iii), take any $i \in \{1, 2\}$, and assume that $H(g_i)$ is not equal to $\{\mathrm{id}_{R[\mathbf{x}]}\}$. Then, g_i satisfies (A) of Theorem 1.2. By (ii) and (i) of Theorem 5.1, g_i also satisfies (B) and (C). Thus, g_i must be of one of the types I through V by Theorem 1.2. Since R is a $\mathbf{Q}$-domain, g_i is not of type II as remarked after Definition 1.1. Since g_i is a coordinate of $R[\mathbf{x}]$ over R, we know that g_i is not of type III by Proposition 2.2. Since $\deg_{x_2} g_2 \geq 2$ by (i), we see that g_2 is not of type I. Therefore, g_i is of type I or IV or V if $i = 1$, and is of type IV or V if $i = 2$. This proves (iii), and thus completing the proof of Theorem 5.1.

Next, we prove Theorem 5.2. Note that $\deg_{x_2} f = 1$ and $I \cap \{1, \ldots, l\} \neq \emptyset$ if and only if $\deg_{h_0} f_0 = 1$ and $t \geq 1$. If these conditions are satisfied, then we see from (2.5) and (5.2) that g_1 has the form $a'(ax_2 - g) + x_1$ for some $a' \in K^\times$ and $g \in K[x_1]$ with $\deg_{x_1} g = t + 1 \geq 2$ whose leading coefficient is equal to $b_t/(t+1)$. Since $b_t/(t+1)$ does not belong to aR, we know that g_1 is of type I. Conversely, if g_1 is of type I, then we see from (2.5) and (5.2) that $\deg_{h_0} f_0 = 1$ and $t \geq 1$. Hence, we have $\deg_{x_2} f = 1$ and $I \cap \{1, \ldots, l\} \neq \emptyset$. This proves (i).

We prove the "only if" part of (ii), and (iii). Take any $i \in \{1, 2\}$, and assume that g_i is of type IV or V. Then, we have $\deg_{x_1} g_i = \deg_{x_2} g_i$ by (1.2). This implies that $t = 0$ by (5.4). Hence, ϕ does not satisfy (W1). Since g_i is a coordinate of $R[\mathbf{x}]$ over R, and is of type IV or V, we know that g_i is wild by the remark before Theorem 1.2. Hence, ϕ does not belong to $\mathrm{T}(R, \mathbf{x})$. Thus, ϕ satisfies (W2). Consequently, ϕ satisfies the first two parts of (5.1). Now, assume that g_i is of type IV. Then, we have $\deg g_i = 2$ by definition. Since $t = 0$, we know by (5.4) that $m = 2$, and $l = 0$ if $i = 2$. Hence, we get $\deg_{x_2} f = 2$ and the last part of (5.1). This proves the "only if" part of (ii). Next, assume that g_i is of type V. Then, we have $\deg g_i \geq 3$ by definition. If $i = 1$, or if $i = 2$ and $l = 0$, then it follows from (5.4) that

$\deg_{x_2} f = m \geq 3$, since $t = 0$. To complete the proof of (iii), it suffices to verify that g_2 is not of type V when $l \geq 1$.

Lemma 5.3. *Assume that f_5 is as in Definition 1.1. Let $\bar{f}_5$ be a linear form in x_1 and x_2 over K such that $f_5^{\mathbf{w}(f_5)} = \alpha \bar{f}_5^u$ for some $\alpha \in K^\times$ and $u \in \mathbf{N}$. Then, we have $d\bar{f}_5 \wedge df_5 = \beta dx_1 \wedge dx_2$ for some $\beta \in K^\times$.*

PROOF. From (1.3), we see that $\bar{f}_5 = c(\alpha_1 x_1 + \alpha_2 x_2)$ for some $c \in K^\times$. Hence, $\tau_5(x_1)$ belongs to $K[\bar{f}_5]$. Since $f = \tau_5(x_2 + g')$ for some $g' \in K[x_1]$, it follows that

$$d\bar{f}_5 \wedge df_5 = d\bar{f}_5 \wedge d\tau(x_2 + g') = d\bar{f}_5 \wedge d\tau_5(x_2) = c(\alpha_1\beta_2 - \alpha_2\beta_1)dx_1 \wedge dx_2.$$

Since τ_5 is an element of $\mathrm{Aff}(K, \mathbf{x})$, we know that $\det J\tau_5 = \alpha_1\beta_2 - \alpha_2\beta_1$ belongs to $K^\times$. Therefore, $c(\alpha_1\beta_2 - \alpha_2\beta_1)$ belongs to $K^\times$. $\square$

Since $t = 0$, we have $I = \{0\}$. Hence, we get $h_0 = a(x_2 - bx_1)$ by (2.5). Thus, we see from (5.3) that $g_2^{\mathbf{w}(g_2)} = ch_0^u$ for some $c \in K^\times$ and $u \in \mathbf{N}$. Now, suppose to the contrary that $l \geq 1$ and g_2 is of type V. Then, we have $dh_0 \wedge dg_2 = \beta dx_1 \wedge dx_2$ for some $\beta \in K^\times$ by Lemma 5.3. By (5.2) with $I = \{0\}$, we get

$$\begin{aligned} dh_0 \wedge dg_2 &= dh_0 \wedge dx_2 + \sum_{i=0}^{l} \frac{b_i}{a}(x_1 + af_0)^i dh_0 \wedge dx_1 - \frac{b_0}{a} dh_0 \wedge dx_1 \\ &= -\sum_{i=0}^{l} b_i (x_1 + af_0)^i dx_1 \wedge dx_2, \end{aligned}$$

since $dh_0 \wedge df_0 = 0$, $dh_0 \wedge dx_1 = -adx_1 \wedge dx_2$ and $dh_0 \wedge dx_2 = -b_0 dx_1 \wedge dx_2$. By the supposition that $l \geq 1$, this implies that $dh_0 \wedge dg_2 \neq \beta dx_1 \wedge dx_2$ for any $\beta \in K^\times$, a contradiction. Therefore, g_2 is not of type V if $l \geq 1$. This completes the proof of Theorem 5.2 (iii).

Finally, we prove the "if" part of Theorem 5.2 (ii). Assume that $\deg_{x_2} f = 2$ and (5.1) is satisfied. We show that g_i is of type IV for $i = 1, 2$ by means of Lemma 4.5 (i). Since (W2) is satisfied, ϕ does not belong to $\mathrm{T}(R, \mathbf{x})$. By Theorems 5.1 (ii), it follows that g_i is tamely reduced over R for $i = 1, 2$. Since $I = \{0\}$ by (5.1), we have $t = 0$. Hence, we get $\deg_{x_1} g_i = \deg_{x_2} g_i$ for $i = 1, 2$ by (5.4). Since $I = \{0\}$, we have $h_0 = ax_2 - b_0 x_1$ by (2.5). Define $\psi \in \mathrm{Aff}(K, \mathbf{x})$ by $\psi(x_1) = h_0$ and $\psi(x_2) = x_1$, and take $q \in K[x_1]$ such that $\psi(q) = f_0$. Then, we have $g_1 = x_1 + af_0 = \psi(x_2 + aq)$ by (5.2), and $\deg_{x_1} q = \deg_{h_0} f_0 = \deg_{x_2} f = 2$ by assumption. Therefore, we conclude that g_1 is of type IV thanks to Lemma 4.5 (i). If $i = 2$, then we have $l = 0$ by the last part of (5.1). Hence, we get $g_2 = x_2 + b_0 f_0$ by (5.2). Define $\psi' \in \mathrm{Aff}(K, \mathbf{x})$ by $\psi'(x_1) = h_0$ and $\psi'(x_2) = x_2$. Then, we have $g_2 = \psi'(x_2 + b_0 q)$. Therefore, we conclude that g_2 is of type IV similarly. This proves the "if" part of (ii) Theorem 5.2, and thus completing the proof of Theorem 5.2.

Theorem 5.4. *For $i \in \{1, 2\}$, the following statements hold*:
(i) *The coordinate $\phi(x_i)$ of $R[\mathbf{x}]$ over R is wild and is not quasi-totally wild if and only if one of the following conditions holds*:
(1) *$i = 1$, $\deg_{x_2} f = 1$ and $I \cap \{1, \ldots, l\} \neq \emptyset$.*

(2) $\deg_{x_2} f = 2$ *and* (*5.1*) *holds.*
(ii) *The coordinate* $\phi(x_i)$ *of* $R[\mathbf{x}]$ *over* R *is quasi-totally wild if and only if one of the following conditions holds*:
(1) $i = 2$ *or* $\deg_{x_2} f \geq 2$, *and* $I \cap \{1, \ldots, l\} \neq \emptyset$.
(2) $i = 2$, $l \geq 1$, $\deg_{x_2} f = 2$, $I = \{0\}$ *and* b_0/a *does not belong to* $V(R)$.
(3) $\deg_{x_2} f \geq 3$, $I = \{0\}$ *and* b_0/a *does not belong to* $V(R)$.
(iii) *If* $I \cap \{1, \ldots, l\} \neq \emptyset$ *and* $\deg_{x_2} f \geq 2$, *then* $\phi(x_1)$ *is a totally wild coordinate of* $R[\mathbf{x}]$ *over* R.
(iv) *If one of the following conditions holds, then* $\phi(x_2)$ *is a totally wild coordinate of* $R[\mathbf{x}]$ *over* R:
(1) $I \cap \{1, \ldots, l\} \neq \emptyset$.
(2) $l \geq 1$, $\deg_{x_2} f \geq 2$, $I = \{0\}$ *and* b_0/a *does not belong to* $V(R)$.

PROOF. We may replace $\phi(x_i)$ with g_i in the statements of the corollary if necessary.

(i) If $\phi(x_i)$ is wild, then ϕ does not belong to $\mathrm{T}(R, \mathbf{x})$. When this is the case, we know by Theorem 5.1 (iii) and Proposition 1.4 that g_i is not quasi-totally wild only if $i = 1$ and g_i is of type I, or g_i is of type IV. Conversely, if g_i is of type I or IV, then g_i is wild and is not quasi-totally wild. Thus, $\phi(x_i)$ is wild and is not quasi-totally wild if and only if $i = 1$ and g_i is of type I, or g_i is of type IV. Therefore, (i) follows from (i) and (ii) of Theorem 5.2.

(ii) Note that $\phi(x_i)$ is quasi-totally wild if and only if $\phi(x_i)$ is wild and (1) and (2) of (i) do not hold. We prove that this condition is equivalent to the condition that one of (1), (2) and (3) of (ii) holds. First, assume that $I \cap \{1, \ldots, l\} \neq \emptyset$. Then, ϕ does not belong to $\mathrm{T}(R, \mathbf{x})$. This implies that $\phi(x_i)$ is wild as mentioned. Since $I \neq \{0\}$, we see that (5.1) does not hold. Hence, (2) of (i) is not satisfied. In this case, (1) of (i) does not hold if and only if $i = 2$ or $\deg_{x_2} f \geq 2$. Hence, g_i is quasi-totally wild if and only if (1) of (ii) holds. Next, assume that $I \cap \{1, \ldots, l\} = \emptyset$. Then, (W1) and (1) of (i) do not hold. Hence, we know that g_i is quasi-totally wild if and only if (W2) holds and (2) of (i) does not hold. Since the first two parts of (W2) and (5.1) are the same, it follows that g_i is quasi-totally wild if and only if (W2) holds, and $\deg_{x_2} f \neq 2$ or $i = 2$ and $l \geq 1$. This condition is equivalent to the condition that (2) or (3) of (ii) is satisfied. Therefore, g_i is quasi-totally wild if and only if one of (1), (2) and (3) of (ii) holds.

(iii) Since $I \cap \{1, \ldots, l\} \neq \emptyset$ by assumption, ϕ does not belong to $\mathrm{T}(R, \mathbf{x})$. Hence, it suffices to show that g_1 is not of type I or IV or V in view of Theorem 5.1 (iii). Since $\deg_{x_2} f \geq 2$ by assumption, g_1 is not of type I by Theorem 5.2 (i). Since $I \neq \{0\}$, we see that (5.1) does not hold. Hence, g_1 is not of type IV or V by (ii) and (iii) of Theorem 5.2. Therefore, g_1 is totally wild.

(iv) Assume that (1) or (2) is satisfied. Then, ϕ does not belong to $\mathrm{T}(R, \mathbf{x})$, since (1) is the same as (W1), and (2) implies (W2). Thus, it suffices to show that g_2 is not of type IV or V in view of Theorem 5.1 (iii). Note that (5.1) does not hold, since $I \neq \{0\}$ in the case of (1), and $l \geq 1$ in the case of (2). Hence, we know by (ii) and (iii) of Theorem 5.2 that g_2 is not of type IV or V. Therefore, g_2 is totally wild. $\square$

Since no elements of $R[\mathbf{x}]$ is of type IV or V when $V(R) = K^{\times}$, Theorem 5.4 implies the following corollary.

Corollary 5.5. *Assume that* $V(R) = K^\times$.
(i) *The following conditions are equivalent*:
(1) ϕ *does not belong to* $\mathrm{T}(R, \mathbf{x})$ *and* $\deg_{x_2} f \geq 2$.
(2) $\phi(x_1)$ *is a quasi-totally wild coordinate of* $R[\mathbf{x}]$ *over* R.
(3) $\phi(x_1)$ *is a totally wild coordinate of* $R[\mathbf{x}]$ *over* R.
(ii) *The following conditions are equivalent*:
(1) ϕ *does not belong to* $\mathrm{T}(R, \mathbf{x})$.
(2) $\phi(x_2)$ *is a wild coordinate of* $R[\mathbf{x}]$ *over* R.
(3) $\phi(x_2)$ *is a quasi-totally wild coordinate of* $R[\mathbf{x}]$ *over* R.
(4) $\phi(x_2)$ *is a totally wild coordinate of* $R[\mathbf{x}]$ *over* R.

PROOF. (i) Assume that (1) is satisfied. From the first part of (1), we have (W1) or (W2). Since $V(R) = K^\times$ by assumption, (W2) does not hold. Hence, we get (W1), and so $I \cap \{1, \ldots, l\} \neq \emptyset$. Since $\deg_{x_2} f \geq 2$ by the last part of (1), we know that $\phi(x_1)$ is totally wild by Theorem 5.4 (iii). Therefore, (1) implies (3). Clearly, (3) implies (2). Assume that $\phi(x_1)$ is quasi-totally wild. Then, one of (1), (2) and (3) of Theorem 5.4 (ii) holds. Since $V(R) = K^\times$ by assumption, (2) and (3) do not hold. Hence, (1) of Theorem 5.4 (ii) holds. Since $i = 1$, it follows that $I \cap \{1, \ldots, l\} \neq \emptyset$ and $\deg_{x_2} f \geq 2$. This implies (1). Therefore, (1), (2) and (3) are equivalent.

(ii) Since $V(R) = K^\times$, we see that (1) implies (W1). By (1) of Theorem 5.4 (iv), (W1) implies (4). Hence, (1) implies (4). Clearly, (4) implies (3), and (2) implies (1). We have shown that (3) implies (2) after Definition 0.1. Therefore, (1) through(4) are equivalent. □

Finally, assume that $n = 3$, and consider the element $\Phi_{g,h}^f = \exp fT_{g,h}$ of $\mathrm{Aut}(k[\mathbf{x}]/k)$ defined before Proposition 2.4 for $(g, h) \in \Lambda$ and $f \in k[x_1, gx_3 + h]$ not belonging to $k[x_1]$. As mentioned, $\Phi_{g,h}^f$ does not belong to $\mathrm{T}(k, \mathbf{x})$. We define

$$H_i = \mathrm{Aut}(k[\mathbf{x}]/k[x_1, \Phi_{g,h}^f(x_i)]) \cap \mathrm{T}(k, \mathbf{x})$$

for $i = 2, 3$. Then, we have the following corollary.

Corollary 5.6. *For* $i \in \{2, 3\}$, *we have* $H_i \neq \{\mathrm{id}_{k[\mathbf{x}]}\}$ *if and only if* $i = 2$ *and* $\deg_{x_3} f = 1$.

PROOF. Let $R = k[x_1]$, $y_i = x_{i+1}$ for $i = 1, 2$ and $\mathbf{y} = \{y_1, y_2\}$, and put $\phi = \Phi_{g,h}^f$. Then, we have $H_i = \mathrm{Aut}(R[\mathbf{y}]/R[\phi(y_{i-1})]) \cap \mathrm{T}(R, \mathbf{y})$ for $i = 2, 3$ on account of Theorem 1. Hence, we have $H_i \neq \{\mathrm{id}_{k[\mathbf{x}]}\}$ if and only if $\phi(y_{i-1})$ is not a totally wild element of $R[\mathbf{y}]$ over R. Note that $T_{g,h}$ is a triangular derivation of $R[\mathbf{y}]$ over R with $T_{g,h}(y_j) = T_{g,h}(x_{j+1}) \neq 0$ for $j = 1, 2$, and ϕ is an element of $\mathrm{Aut}(R[\mathbf{y}]/R)$ not belonging to $\mathrm{T}(R, \mathbf{y})$. Hence, $\phi(y_1)$ is totally wild if and only if $\deg_{y_2} f \geq 2$ by Corollary 5.5 (i), and $\phi(y_2)$ is always totally wild by Corollary 5.5 (ii). Since f is not an element of $k[x_1]$, we have $\deg_{y_2} f = \deg_{x_3} f \geq 1$. Thus, $\phi(y_{i-1})$ is not totally wild if and only if $i = 2$ and $\deg_{y_2} f = 1$. Therefore, we have $H_i \neq \{\mathrm{id}_{k[\mathbf{x}]}\}$ if and only if $i = 2$ and $\deg_{x_3} f = 1$. □

In Chapter 6, we will give coordinates of $k[\mathbf{x}]$ over k some of which are totally wild, and others are quasi-totally wild, but not totally wild.

Part 2

Applications of the generalized Shestakov-Umirbaev theory

CHAPTER 5

Generalized Shestakov-Umirbaev theory

1. Shestakov-Umirbaev reductions

Part 2 is devoted to applications of the generalized Shestakov-Umirbaev theory. Throughout, a *tame* (resp. *wild*) automorphism of $k[\mathbf{x}]$ will always mean an element of $\mathrm{T}(k,\mathbf{x})$ (resp. $\mathrm{Aut}(k[\mathbf{x}]/k)\setminus \mathrm{T}(k,\mathbf{x})$). In this chapter, we briefly review the generalized Shestakov-Umirbaev theory, and derive some consequences needed later.

Let Γ be a totally ordered additive group, and let $F=(f_1,f_2,f_3)$ and $G=(g_1,g_2,g_3)$ be triples of elements of $k[\mathbf{x}]$ such that f_1, f_2, f_3 and g_1, g_2, g_3 are algebraically independent over k, respectively. Here, $n\in\mathbf{N}$ may be arbitrary for the moment. We denote by $\Gamma_{>0}$ the set of positive elements of Γ. For $\mathbf{w}\in(\Gamma_{>0})^n$, we say that the pair (F,G) satisfies the *Shestakov-Umirbaev condition* for the weight $\mathbf{w}$ if the following conditions hold (cf. [**16**]):

(SU1) $g_1=f_1+af_3^2+cf_3$ and $g_2=f_2+bf_3$ for some $a,b,c\in k$, and g_3-f_3 belongs to $k[g_1,g_2]$;
(SU2) $\deg_{\mathbf{w}} f_1\le\deg_{\mathbf{w}} g_1$ and $\deg_{\mathbf{w}} f_2=\deg_{\mathbf{w}} g_2$;
(SU3) $(g_1^{\mathbf{w}})^2\approx(g_2^{\mathbf{w}})^s$ for some odd number $s\ge 3$;
(SU4) $\deg_{\mathbf{w}} f_3\le\deg_{\mathbf{w}} g_1$, and $f_3^{\mathbf{w}}$ does not belong to $k[g_1^{\mathbf{w}},g_2^{\mathbf{w}}]$;
(SU5) $\deg_{\mathbf{w}} g_3<\deg_{\mathbf{w}} f_3$;
(SU6) $\deg_{\mathbf{w}} g_3<\deg_{\mathbf{w}} g_1-\deg_{\mathbf{w}} g_2+\deg_{\mathbf{w}} dg_1\wedge dg_2$.

Here, $h_1\approx h_2$ (resp. $h_1\not\approx h_2$) denotes that h_1 and h_2 are linearly dependent (resp. linearly independent) over k for each $h_1,h_2\in k[\mathbf{x}]\setminus\{0\}$. We say that (F,G) satisfies the *weak Shestakov-Umirbaev condition* for the weight $\mathbf{w}$ if (SU4), (SU5), (SU6) and the following conditions are satisfied (cf. [**16**]):

(SU1$'$) g_1-f_1, g_2-f_2 and g_3-f_3 belong to $k[f_2,f_3]$, $k[f_3]$ and $k[g_1,g_2]$, respectively;
(SU2$'$) $\deg f_i\le\deg g_i$ for $i=1,2$;
(SU3$'$) $\deg g_2<\deg g_1$, and $g_1^{\mathbf{w}}$ does not belong to $k[g_2^{\mathbf{w}}]$.

It is easy to check that (SU1), (SU2) and (SU3) imply (SU1$'$), (SU2$'$) and (SU3$'$), respectively. Hence, the Shestakov-Umirbaev condition implies the weak Shestakov-Umirbaev condition. As listed in [**16**, Theorem 4.2], if (F,G) satisfies the weak Shestakov-Umirbaev condition for the weight $\mathbf{w}$, then (F,G) has certain special properties such as

(P1) $(g_1^{\mathbf{w}})^2\approx(g_2^{\mathbf{w}})^s$ for some odd number $s\ge 3$, and so $\delta:=(1/2)\deg_{\mathbf{w}} g_2$ belongs to Γ.

(P2) $\deg_{\mathbf{w}} f_3\ge(s-2)\delta+\deg_{\mathbf{w}} dg_1\wedge dg_2$.

(P5) If $\deg_{\mathbf{w}} f_1 < \deg_{\mathbf{w}} g_1$, then $s = 3$, $g_1^{\mathbf{w}} \approx (f_3^{\mathbf{w}})^2$, $\deg_{\mathbf{w}} f_3 = (3/2)\delta$ and

$$\deg_{\mathbf{w}} f_1 \geq \frac{5}{2}\delta + \deg_{\mathbf{w}} dg_1 \wedge dg_2.$$

(P7) $\deg_{\mathbf{w}} f_2 < \deg_{\mathbf{w}} f_1$, $\deg_{\mathbf{w}} f_3 \leq \deg_{\mathbf{w}} f_1$, and $\delta < \deg_{\mathbf{w}} f_i \leq s\delta$ for $i = 1, 2, 3$.

In what follows, we simply say that elements of Γ are linearly dependent (resp. linearly independent) if they are linearly dependent (resp. linearly independent) over $\mathbf{Z}$.

Lemma 1.1. *Assume that (F, G) satisfies the Shestakov-Umirbaev condition for the weight $\mathbf{w}$. Then, $\deg_{\mathbf{w}} f_i$ and $\deg_{\mathbf{w}} f_2$ are linearly dependent for $i = 1$ or $i = 3$.*

Proof. If $\deg_{\mathbf{w}} f_1 = \deg_{\mathbf{w}} g_1$, then we have

$$2\deg_{\mathbf{w}} f_1 = 2\deg_{\mathbf{w}} g_1 = s\deg_{\mathbf{w}} g_2 = s\deg_{\mathbf{w}} f_2$$

for some odd number $s \geq 3$ by (SU3) and (SU2). Hence, $\deg_{\mathbf{w}} f_1$ and $\deg_{\mathbf{w}} f_2$ are linearly dependent. If $\deg_{\mathbf{w}} f_1 \neq \deg_{\mathbf{w}} g_1$, then we have $\deg_{\mathbf{w}} f_1 < \deg_{\mathbf{w}} g_1$ by (SU2), and hence $\deg_{\mathbf{w}} f_3 = (3/2)\delta = (3/4)\deg_{\mathbf{w}} g_2$ by (P5). Since $\deg_{\mathbf{w}} g_2 = \deg_{\mathbf{w}} f_2$ by (SU2), it follows that $\deg_{\mathbf{w}} f_3$ and $\deg_{\mathbf{w}} f_2$ are linearly dependent. $\square$

We define the *rank* $\operatorname{rank}\mathbf{w}$ of $\mathbf{w} = (w_1, \ldots, w_n)$ as the rank of the $\mathbf{Z}$-submodule of Γ generated by $w_1, \ldots, w_n$. If $\operatorname{rank}\mathbf{w} = n$, then $x_1^{a_1} \cdots x_n^{a_n}$'s have the different $\mathbf{w}$-degrees for different $(a_1, \ldots, a_n)$'s. Hence, $f^{\mathbf{w}}$ and $g^{\mathbf{w}}$ are monomials for each $f, g \in k[\mathbf{x}] \setminus \{0\}$. When this is the case, $f^{\mathbf{w}}$ and $g^{\mathbf{w}}$ are algebraically independent over k if and only if $\deg_{\mathbf{w}} f$ and $\deg_{\mathbf{w}} g$ are linearly independent.

Now, assume that $n = 3$. Then, we may identify $F \in \operatorname{Aut}(k[\mathbf{x}]/k)$ with the triple (f_1, f_2, f_3), where $f_i := F(x_i)$ for $i = 1, 2, 3$. For a permutation σ of $\{1, 2, 3\}$, we define $F_\sigma = (f_{\sigma(1)}, f_{\sigma(2)}, f_{\sigma(3)})$. We say that F admits a *Shestakov-Umirbaev reduction* for the weight $\mathbf{w}$ if there exist a permutation σ of $\{1, 2, 3\}$ and $G \in \operatorname{Aut}(k[\mathbf{x}]/k)$ such that (F_σ, G_σ) satisfies the Shestakov-Umirbaev condition for the weight $\mathbf{w}$. If this is the case, $\deg_{\mathbf{w}} f_i$ and $\deg_{\mathbf{w}} f_j$ must be linearly dependent for some $i \neq j$ by virtue of Lemma 1.1. If $\operatorname{rank}\mathbf{w} = 3$, then this implies that $f_i^{\mathbf{w}}$ and $f_j^{\mathbf{w}}$ are algebraically dependent over k for some $i \neq j$. Therefore, if $\operatorname{rank}\mathbf{w} = 3$, and $f_1^{\mathbf{w}}$, $f_2^{\mathbf{w}}$ and $f_3^{\mathbf{w}}$ are pairwise algebraically independent over k, then F admits no Shestakov-Umirbaev reduction for the weight $\mathbf{w}$.

Lemma 1.2. *Assume that $\deg_{\mathbf{w}} f_1 > \deg_{\mathbf{w}} f_2 > \deg_{\mathbf{w}} f_3$. If F admits a Shestakov-Umirbaev reduction for the weight $\mathbf{w}$, then we have $3\deg_{\mathbf{w}} f_2 = 4\deg_{\mathbf{w}} f_3$, or $2\deg_{\mathbf{w}} f_1 = s\deg_{\mathbf{w}} f_i$ for some odd number $s \geq 3$ and $i \in \{2, 3\}$.*

Proof. By definition, there exist σ and G such that (F_σ, G_σ) satisfies the Shestakov-Umirbaev condition for the weight $\mathbf{w}$. Since $\deg_{\mathbf{w}} f_1 > \deg_{\mathbf{w}} f_i$ for $i = 2, 3$ by assumption, we know that $\sigma(1) = 1$ in view of (P7). If $\deg_{\mathbf{w}} f_1 = \deg_{\mathbf{w}} g_1$, then we have $2\deg_{\mathbf{w}} f_1 = s\deg_{\mathbf{w}} f_{\sigma(2)}$ for some odd number $s \geq 3$ by (SU3) and (SU2). Since $\sigma(2)$ must be 2 or 3, we

get the last statement. If $\deg_{\mathbf{w}} f_1 \neq \deg_{\mathbf{w}} g_1$, then we have $\deg_{\mathbf{w}} f_{\sigma(3)} = (3/2)(1/2)\deg_{\mathbf{w}} f_{\sigma(2)}$ by (P5) and (SU2). Hence, we get $3\deg_{\mathbf{w}} f_{\sigma(2)} = 4\deg_{\mathbf{w}} f_{\sigma(3)}$. Since $\deg f_2 > \deg_{\mathbf{w}} f_3$ by assumption, it follows that σ is the identity permutation. Therefore, we obtain $3\deg_{\mathbf{w}} f_2 = 4\deg_{\mathbf{w}} f_3$. $\square$

The following theorem is a generalization of the main result of Shestakov-Umirbaev [**27**].

Theorem 1.3 ([**16**, Theorem 2.1])**.** *Assume that* $n = 3$*. If* $\deg_{\mathbf{w}} \phi > |\mathbf{w}|$ *for* $\phi \in \mathrm{T}(k, \mathbf{x})$ *and* $\mathbf{w} \in (\Gamma_{>0})^3$*, then* ϕ *admits an elementary reduction for the weight* $\mathbf{w}$*, or a Shestakov-Umirbaev reduction for the weight* $\mathbf{w}$*.*

We mention that $\phi \in \mathrm{Aut}(k[\mathbf{x}]/k)$ is tame if $\deg_{\mathbf{w}} \phi = |\mathbf{w}|$ (cf. [**16**, Lemma 6.1]). By Lemma 1.1, we have $\deg_{\mathbf{w}} \phi = |\mathbf{w}|$ if and only if $\phi(x_1)^{\mathbf{w}}$, $\phi(x_2)^{\mathbf{w}}$ and $\phi(x_3)^{\mathbf{w}}$ are algebraically independent over k.

By Theorem 1.3, it follows that $F = (f_1, f_2, f_3) \in \mathrm{Aut}(k[\mathbf{x}]/k)$ is wild if there exists $\mathbf{w} \in (\Gamma_{>0})^3$ with $\operatorname{rank} \mathbf{w} = 3$ as follows:

(1) $f_1^{\mathbf{w}}$, $f_2^{\mathbf{w}}$ and $f_3^{\mathbf{w}}$ are algebraically dependent over k, and are pairwise algebraically independent over k;

(2) $f_i^{\mathbf{w}}$ does not belong to $k[\{f_j^{\mathbf{w}} \mid j \neq i\}]$ for $i = 1, 2, 3$.

In fact, the former part of (1) implies that $\deg_{\mathbf{w}} F > |\mathbf{w}|$. Since $\operatorname{rank} \mathbf{w} = 3$, the latter part of (1) implies that F admits no Shestakov-Umirbaev reduction for the weight $\mathbf{w}$ as mentioned. The latter part of (1) also implies that $k[f_i, f_j]^{\mathbf{w}} = k[f_i^{\mathbf{w}}, f_j^{\mathbf{w}}]$ for each $i \neq j$ by the discussion before Lemma 1.1. Hence, (2) implies that F admits no elementary reduction for the weight $\mathbf{w}$.

Definition 1.1. We call $P \in k[\mathbf{x}]$ a *W-test polynomial* if there do not exist $\phi \in \mathrm{T}(k, \mathbf{x})$, totally ordered additive group Γ and $\mathbf{w} \in (\Gamma_{>0})^3$ with $\operatorname{rank} \mathbf{w} = 3$ which satisfy the following conditions:

(a) $\deg_{\mathbf{w}} \phi(P) < \deg_{\mathbf{w}} \phi(x_{i_1})$ for some $i_1 \in \{1, 2, 3\}$;

(b) $\deg_{\mathbf{w}} \phi(x_{i_2})$ and $\deg_{\mathbf{w}} \phi(x_{i_3})$ are linearly independent for some $i_2, i_3 \in \{1, 2, 3\}$.

Since $\operatorname{rank} \mathbf{w} = 3$, the condition (b) is equivalent to the condition that $\phi(x_{i_2})^{\mathbf{w}}$ and $\phi(x_{i_3})^{\mathbf{w}}$ are algebraically independent over k for some $i_2, i_3 \in \{1, 2, 3\}$. Note that P is a W-test polynomial if and only if the following condition holds:

(†) If $\phi \in \mathrm{Aut}(k[\mathbf{x}]/k)$ satisfies (a) and (b) for some totally ordered additive group Γ and $\mathbf{w} \in (\Gamma_{>0})^3$ with $\operatorname{rank} \mathbf{w} = 3$, then ϕ does not belong to $\mathrm{T}(k, \mathbf{x})$.

For $P \in k[\mathbf{x}]$, we define

$$\mathcal{F}(P) = \{P^{\mathbf{v}} \mid \mathbf{v} \in (\Lambda_{>0})^3,\ \Lambda \text{ is a totally ordered additive group}\}.$$

The following result will be used in Chapter 7 to prove the wildness of certain exponential automorphisms.

Proposition 1.4. *Assume that* $P \in k[\mathbf{x}]$ *does not belong to* $k[\mathbf{x} \setminus \{x_i\}]$ *for* $i = 1, 2, 3$*. If the following conditions hold for each* $f \in \mathcal{F}(P)$*, then* P *is a W-test polynomial*:

(i) f *is not divisible by* $x_i - g$ *for any* $i \in \{1, 2, 3\}$ *and* $g \in k[\mathbf{x} \setminus \{x_i\}] \setminus k$*.*

(ii) f *is not divisible by* $x_i^{s_i} - c x_j^{s_j}$ *for any* $i, j \in \{1, 2, 3\}$ *with* $i \neq j$*,* $s_i, s_j \in \mathbf{N}$ *and* $c \in k^{\times}$*.*

Proof. Let P be as in the proposition. We verify that P satisfies (†). Assume that $\phi \in \mathrm{Aut}(k[\mathbf{x}]/k)$ satisfies (a) and (b) for some totally ordered additive group Γ and $\mathbf{w} \in (\Gamma_{>0})^3$ with $\operatorname{rank} \mathbf{w} = 3$. Then, we show that ϕ is wild. Since $\operatorname{rank} \mathbf{w} = 3$, it suffices to check the conditions (1) and (2) after Theorem 1.3. Set $f_i = \phi(x_i)$ and $v_i = \deg_{\mathbf{w}} f_i$ for $i = 1, 2, 3$. Then, $\mathbf{v} := (v_1, v_2, v_3)$ belongs to $(\Gamma_{>0})^3$, since so does $\mathbf{w}$, and f_1, f_2 and f_3 are not constants. Because P does not belong to $k[\mathbf{x} \setminus \{x_i\}]$ for $i = 1, 2, 3$ by assumption, there appears in P a monomial involving x_{i_1}. Hence, we have $\deg_{\mathbf{v}} P \geq v_{i_1}$. By (a), $v_{i_1} = \deg_{\mathbf{w}} \phi(x_{i_1})$ is greater than $\deg_{\mathbf{w}} \phi(P)$. Thus, we get $\deg_{\mathbf{v}} P > \deg_{\mathbf{w}} \phi(P)$. This implies that $\psi(P^{\mathbf{v}}) = 0$, where ψ is the endomorphism of the k-algebra $k[\mathbf{x}]$ defined by $\psi(x_i) = f_i^{\mathbf{w}}$ for $i = 1, 2, 3$. Consequently, we know that $f_1^{\mathbf{w}}$, $f_2^{\mathbf{w}}$ and $f_3^{\mathbf{w}}$ are algebraically dependent over k.

We show that $f_1^{\mathbf{w}}$, $f_2^{\mathbf{w}}$ and $f_3^{\mathbf{w}}$ are pairwise algebraically independent over k by contradiction. Suppose that $f_i^{\mathbf{w}}$ and $f_j^{\mathbf{w}}$ are algebraically dependent over k for some $i \neq j$, say $i = 2$ and $j = 3$. Since $\operatorname{rank} \mathbf{w} = 3$ by assumption, $f_2^{\mathbf{w}}$ and $f_3^{\mathbf{w}}$ are monomials. Hence, we have $(f_3^{\mathbf{w}})^s = c(f_2^{\mathbf{w}})^t$ for some $c \in k^{\times}$ and $s, t \in \mathbf{N}$ with $\gcd(s, t) = 1$. Then, $x_3^s - cx_2^t$ is an irreducible element of $k[\mathbf{x}]$. Since $x_3^s - cx_2^t$ is a monic polynomial in x_3, it follows that $x_3^s - cx_2^t$ is an irreducible polynomial in x_3 over $k[x_1, x_2]$, and hence over $k(x_1, x_2)$. Note that $f_1^{\mathbf{w}}$ and $f_2^{\mathbf{w}}$ are algebraically independent over k by (b), since $f_2^{\mathbf{w}}$ and $f_3^{\mathbf{w}}$ are algebraically dependent over k by supposition. Let z be an indeterminate over $k[\mathbf{x}]$. Then, $f_1^{\mathbf{w}}$, $f_2^{\mathbf{w}}$ and z are algebraically independent over k. Hence, $p(z) := z^s - c(f_2^{\mathbf{w}})^t$ is an irreducible polynomial in z over $k(f_1^{\mathbf{w}}, f_2^{\mathbf{w}})$. Since $p(f_3^{\mathbf{w}}) = (f_3^{\mathbf{w}})^s - c(f_2^{\mathbf{w}})^t = 0$, it follows that $p(z)$ is the minimal polynomial of $f_3^{\mathbf{w}}$ over $k(f_1^{\mathbf{w}}, f_2^{\mathbf{w}})$. Let $q(z)$ be the element of $k[f_1^{\mathbf{w}}, f_2^{\mathbf{w}}][z]$ obtained from $P^{\mathbf{v}}$ by the substitution $x_i \mapsto f_i^{\mathbf{w}}$ for $i = 1, 2$ and $x_3 \mapsto z$. Then, we have $q(f_3^{\mathbf{w}}) = \psi(P^{\mathbf{v}}) = 0$. Hence, $q(z)$ is divisible by $p(z)$. Accordingly, $P^{\mathbf{v}}$ is divisible by $x_3^s - cx_2^t$, a contradiction to (ii). Thus, $f_1^{\mathbf{w}}$, $f_2^{\mathbf{w}}$ and $f_3^{\mathbf{w}}$ are pairwise algebraically independent over k, proving (1). As a consequence, we know that $\ker \psi$ is a prime ideal of $k[\mathbf{x}]$ of height one, and hence a principal ideal of $k[\mathbf{x}]$.

Finally, we show (2) by contradiction. Suppose to the contrary that $f_i^{\mathbf{w}}$ belongs to $k[\{f_j^{\mathbf{w}} \mid j \neq i\}]$ for some i, say $i = 1$. Then, there exists $g \in k[x_2, x_3] \setminus k$ such that $f_1^{\mathbf{w}} = \psi(g)$. Note that $x_1 - g$ is an irreducible element of $k[\mathbf{x}]$ such that $\psi(x_1 - g) = f_1^{\mathbf{w}} - \psi(g) = 0$. Since $\ker \psi$ is a principal prime ideal of $k[\mathbf{x}]$ as mentioned, this implies that $\ker \psi$ is generated by $x_1 - g$. Since $P^{\mathbf{v}}$ belongs to $\ker \psi$, it follows that $P^{\mathbf{v}}$ is divisible by $x_1 - g$. This contradicts (i). Therefore, ϕ satisfies (2). This proves that ϕ is wild, and thereby proving that P is a W-test polynomial. □

2. Shestakov-Umirbaev inequality

In this section, we review the generalized Shestakov-Umirbaev inequality [**15**]. Consider a nonzero polynomial Φ in one variable z over $k[\mathbf{x}]$. Take $\mathbf{w} \in (\Gamma_{>0})^n$ and $g \in k[\mathbf{x}] \setminus \{0\}$, and set $\mathbf{w}_g = (\mathbf{w}, \deg_{\mathbf{w}} g)$. Regard Φ as a polynomial in $n + 1$ variables over k with $x_{n+1} = z$. Then, we have

$$\deg_{\mathbf{w}}^g \Phi := \deg_{\mathbf{w}_g} \Phi \geq \deg_{\mathbf{w}} \Phi(g), \quad \deg_{\mathbf{w}}^g \Phi \geq (\deg_z \Phi) \deg_{\mathbf{w}} g.$$

We denote by $\Phi^{(i)}$ the i-th order derivative of Φ in z for each $i \geq 0$. Then, we have $\deg_{\mathbf{w}}^{g} \Phi^{(i)} = \deg_{\mathbf{w}} \Phi^{(i)}(g)$ for sufficiently large $i \geq 0$. We define $m_{\mathbf{w}}^{g}(\Phi)$ to be the minimal $i \in \mathbf{Z}_{\geq 0}$ such that $\deg_{\mathbf{w}}^{g} \Phi^{(i)} = \deg_{\mathbf{w}} \Phi^{(i)}(g)$. We mention that $m_{\mathbf{w}}^{g}(\Phi)$ is equal to the minimal $i \in \mathbf{Z}_{\geq 0}$ such that $(\Phi^{\mathbf{w}_g})^{(i)}(g^{\mathbf{w}}) \neq 0$ (see [**15**, Lemma 3.1(ii)]).

With this notation, we have the following theorem. This is a generalization of Shestakov-Umirbaev [**26**, Theorem 3].

Theorem 2.1 ([**15**, Theorem 2.1])**.** *Let $f_1, \ldots, f_r$ be elements of $k[\mathbf{x}]$ which are algebraically independent over k, and $\omega := df_1 \wedge \cdots \wedge df_r$, where $r \in \mathbf{N}$. If Φ belongs to $k[f_1, \ldots, f_r][z] \setminus \{0\}$, then we have*

$$\deg_{\mathbf{w}} \Phi(g) \geq \deg_{\mathbf{w}}^{g} \Phi + m_{\mathbf{w}}^{g}(\Phi)(\deg_{\mathbf{w}} \omega \wedge dg - \deg_{\mathbf{w}} \omega - \deg_{\mathbf{w}} g).$$

In the situation of Theorem 2.1, $\Phi^{\mathbf{w}_g}$ belongs to $k[f_1, \ldots, f_r]^{\mathbf{w}}[z] \setminus \{0\}$. Let K be the field of fractions of $k[f_1, \ldots, f_r]^{\mathbf{w}}$, $\psi(z)$ the minimal polynomial of $g^{\mathbf{w}}$ over K, and $m := m_{\mathbf{w}}^{g}(\Phi)$. Then, $\Phi^{\mathbf{w}_g}$ is divisible by $\psi(z)^m$, since m is equal to the minimal number such that $(\Phi^{\mathbf{w}_g})^{(m)}(g^{\mathbf{w}}) \neq 0$ as mentioned. Since $\deg_z \Phi^{\mathbf{w}_g} \leq \deg_z \Phi$, it follows that m is at most the quotient of $\deg_z \Phi$ divided by $[K(g^{\mathbf{w}}) : K]$.

Now, let $S = \{f, g\} \subset k[\mathbf{x}]$ be such that f and g are algebraically independent over k, and ϕ a nonzero element of $k[f, g]$. Then, we may uniquely write $\phi = \sum_{i,j} c_{i,j} f^i g^j$, where $c_{i,j} \in k$ for each $i, j \in \mathbf{Z}_{\geq 0}$. We define $\deg_{\mathbf{w}}^{S} \phi$ to be the maximum among $\deg_{\mathbf{w}} f^i g^j$ for $i, j \in \mathbf{Z}_{\geq 0}$ with $c_{i,j} \neq 0$. We remark that, if $f^{\mathbf{w}}$ and $g^{\mathbf{w}}$ are algebraically independent over k, then $\deg_{\mathbf{w}}^{S} \phi$ is equal to $\deg_{\mathbf{w}} \phi$. Take $\Phi \in k[f][y]$ such that $\Phi(g) = \phi$. Then, we have $\deg_{\mathbf{w}}^{g} \Phi = \deg_{\mathbf{w}}^{S} \phi$. Hence, it follows that $\deg_{\mathbf{w}} \phi < \deg_{\mathbf{w}}^{S} \phi$ if and only if $m_{\mathbf{w}}^{g}(\Phi) \geq 1$.

Lemma 2.2. *If $\deg_{\mathbf{w}} \phi < \deg_{\mathbf{w}}^{S} \phi$ for $\phi \in k[f, g] \setminus \{0\}$, then the following assertions hold*:
(i) *There exist $p, q \in \mathbf{N}$ with $\gcd(p, q) = 1$ such that $(g^{\mathbf{w}})^p \approx (f^{\mathbf{w}})^q$.*
(ii) $\deg_{\mathbf{w}} \phi \geq q \deg_{\mathbf{w}} f + \deg_{\mathbf{w}} df \wedge dg - \deg_{\mathbf{w}} f - \deg_{\mathbf{w}} g$.
(iii) *Assume that $\deg_{\mathbf{w}} f < \deg_{\mathbf{w}} g$, $\deg_{\mathbf{w}} \phi \leq \deg_{\mathbf{w}} g$ and $g^{\mathbf{w}}$ does not belong to $k[f^{\mathbf{w}}]$. Then, we have $p = 2$, and $q \geq 3$ is an odd number. Moreover, $\delta := (1/2) \deg_{\mathbf{w}} f$ belongs to Γ, and*

$$\deg_{\mathbf{w}} \phi \geq (q-2)\delta + \deg_{\mathbf{w}} df \wedge dg > \deg_{\mathbf{w}} g - \deg_{\mathbf{w}} f.$$

If furthermore $\deg_{\mathbf{w}} \phi \leq \deg_{\mathbf{w}} f$, then we have $q = 3$.
(iv) *Let $\Phi \in k[\mathbf{x}][z]$ be such that $\Phi(g) = \phi$. Then, $m_{\mathbf{w}}^{g}(\Phi)$ is at most the quotient of $\deg_z \Phi$ divided by p.*

Proof. (i) and (ii), and (iii) follow from Lemmas 3.2 and 3.3 of [**16**], respectively. We prove (iv). Since $k[f]^{\mathbf{w}} = k[f^{\mathbf{w}}]$, the field of fractions of $k[f]^{\mathbf{w}}$ is equal to $k(f^{\mathbf{w}})$. Hence, $m_{\mathbf{w}}^{g}(\Phi)$ is at most the quotient of $\deg_z \Phi$ divided by $[k(f^{\mathbf{w}})(g^{\mathbf{w}}) : k(f^{\mathbf{w}})]$ as remarked above. Since $(g^{\mathbf{w}})^p \approx (f^{\mathbf{w}})^q$, there exists $c \in k^{\times}$ such that $(g^{\mathbf{w}})^p = c(f^{\mathbf{w}})^q$. Then, $z^p - c(f^{\mathbf{w}})^q$ is the minimal polynomial of $g^{\mathbf{w}}$ over $k(f^{\mathbf{w}})$, since $\gcd(p, q) = 1$. Hence, we have $[k(f^{\mathbf{w}})(g^{\mathbf{w}}) : k(f^{\mathbf{w}})] = p$. Therefore, $m_{\mathbf{w}}^{g}(\Phi)$ is at most the quotient of $\deg_z \Phi$ divided by p. $\square$

Using the results above, we prove a technical lemma which will be used in Chapters 6 and 7. Assume that $f, g \in k[\mathbf{x}]$ satisfy the following conditions:

(1) $\deg_{\mathbf{w}} f < \deg_{\mathbf{w}} g$;
(2) $g^{\mathbf{w}}$ does not belong to $k[f^{\mathbf{w}}]$;
(3) f and g are algebraically independent over k.

Then, we define

$$\eta_1 = \deg_{\mathbf{w}} f + \frac{3}{2}\deg_{\mathbf{w}} g \quad \text{and} \quad \eta_2 = 2\deg_{\mathbf{w}} f + \deg_{\mathbf{w}} g.$$

Take any $\theta(z) \in k[z]$ with $d := \deg_z \theta(z) \geq 1$. Then, there exists

$$\eta(\theta; f, g) := \min\{\deg_{\mathbf{w}}(\theta(g) + fh) \mid h \in k[f, g]\},$$

since $\{\deg_{\mathbf{w}} h \mid h \in k[\mathbf{x}]\setminus\{0\}\}$ is a well-ordered subset of Γ by the assumption that $\mathbf{w}$ is an element of $(\Gamma_{>0})^3$ (cf. [**16**, Lemma 6.1]).

With the notation and assumption above, we have the following lemma.

Lemma 2.3. *If one of the following three conditions is satisfied, then we have $\eta(\theta; f, g) > \eta_i$ for $i = 1, 2$:*
(i) *$d = 2$, $\deg_{\mathbf{w}} df \wedge dg > \deg_{\mathbf{w}} g$ and $(2l + 1)\deg_{\mathbf{w}} f = l \deg_{\mathbf{w}} g$ for some integer $l \geq 3$.*
(ii) *$d \geq 3$ and $\deg_{\mathbf{w}} df \wedge dg > (d - 1)\deg_{\mathbf{w}} f$.*
(iii) *$d \geq 9$ and $d \neq 10, 12$.*

Proof. Take $h \in k[f, g]$ and $\Phi \in k[f][z]$ such that $\eta(\theta; f, g) = \deg_{\mathbf{w}}(\theta(g) + fh)$ and $h = \Phi(g)$, and put $\Psi = \theta + f\Phi$. Then, we have $\eta(\theta; f, g) = \deg_{\mathbf{w}} \Psi(g)$. Note that

$$\deg_z \Psi = \max\{\deg_z \theta, \deg_z f\Phi\} \geq \deg_z \theta = d, \tag{2.1}$$

since the leading coefficient of θ is an element of $k^\times$, while that of $f\Phi$ is a multiple of f. Hence, we know that

$$\deg_{\mathbf{w}}^g \Psi = \deg_{\mathbf{w}_g} \Psi \geq (\deg_z \Psi)\deg_{\mathbf{w}} g \geq d \deg_{\mathbf{w}} g. \tag{2.2}$$

First, assume that $m_{\mathbf{w}}^g(\Psi) = 0$. Then, we have $\deg_{\mathbf{w}}^g \Psi = \deg_{\mathbf{w}} \Psi(g)$. Since $\eta(\theta; f, g) = \deg_{\mathbf{w}} \Psi(g)$, we get $\eta(\theta; f, g) \geq d \deg_{\mathbf{w}} g$ by (2.2). Assume that (i) is satisfied. Then, it follows that $\eta(\theta; f, g) \geq 2\deg_{\mathbf{w}} g$, since $d = 2$. Because

$$\deg_{\mathbf{w}} f = \frac{l}{2l + 1}\deg_{\mathbf{w}} g$$

for some $l \geq 3$, we know that $\deg_{\mathbf{w}} f$ is less than $(1/2)\deg_{\mathbf{w}} g$. Hence, we see that $\eta_i < 2\deg_{\mathbf{w}} g$ for $i = 1, 2$. Therefore, we get $\eta(\theta; f, g) > \eta_i$ for $i = 1, 2$. If (ii) or (iii) is satisfied, then we have $\eta(\theta; f, g) \geq 3\deg_{\mathbf{w}} g$ for $i = 1, 2$, since $d \geq 3$. Because $\deg_{\mathbf{w}} f < \deg_{\mathbf{w}} g$ by (1), we see that $\eta_i < 3\deg_{\mathbf{w}} g$ for $i = 1, 2$. Therefore, we get $\eta(\theta; f, g) > \eta_i$ for $i = 1, 2$.

Next, assume that $m_{\mathbf{w}}^g(\Psi) \geq 1$. Then, $\deg_{\mathbf{w}} \Psi(g)$ is less than $\deg_{\mathbf{w}}^g \Psi = \deg_{\mathbf{w}}^S \Psi(g)$, where $S := \{f, g\}$. By Lemma 2.2 (i), there exist $p, q \in \mathbf{N}$ with $\gcd(p, q) = 1$ such that $(g^{\mathbf{w}})^p \approx (f^{\mathbf{w}})^q$. Then, we have $2 \leq p < q$ by (1) and (2). Let a and b be the quotient and remainder of $\deg_z \Psi$ divided by p. Then, we have $a \geq m_{\mathbf{w}}^g(\Psi)$ by Lemma 2.2 (iv). Since $m_{\mathbf{w}}^g(\Psi) \geq 1$ by

assumption, it follows that $a \geq 1$. Set $\delta = p^{-1} \deg_{\mathbf{w}} f$. Then, we have $\deg_{\mathbf{w}} f = p\delta$, $\deg_{\mathbf{w}} g = q\delta$ and

$$\eta_1 = \left(p + \frac{3}{2}q\right)\delta, \quad \eta_2 = (2p+q)\delta. \tag{2.3}$$

Since f does not belong to k by (3), and $\mathbf{w}$ is an element of $(\Gamma_{>0})^3$, we have $\deg_{\mathbf{w}} df = \deg_{\mathbf{w}} f$ by (1.2) and the note following it. Hence, we get

$$\eta(\theta; f, g) = \deg_{\mathbf{w}} \Psi(g) \geq \deg_{\mathbf{w}}^g \Psi + m_{\mathbf{w}}^g(\Psi)(\deg_{\mathbf{w}} df \wedge dg - \deg_{\mathbf{w}} f - \deg_{\mathbf{w}} g)$$

by Theorem 2.1. Since $m_{\mathbf{w}}^g(\Psi) \leq a$, and $\deg_{\mathbf{w}} df \wedge dg \leq \deg_{\mathbf{w}} f + \deg_{\mathbf{w}} g$ by (1.3), we have

$$\begin{aligned} m_{\mathbf{w}}^g(\Psi)(\deg_{\mathbf{w}} df \wedge dg - \deg_{\mathbf{w}} f - \deg_{\mathbf{w}} g) &\geq a(\deg_{\mathbf{w}} df \wedge dg - \deg_{\mathbf{w}} f - \deg_{\mathbf{w}} g) \\ &= a(\deg_{\mathbf{w}} df \wedge dg - p\delta - q\delta). \end{aligned}$$

By (2.2), we know that $\deg_{\mathbf{w}}^g \Psi \geq (ap+b)q\delta$. Therefore, we get

$$\begin{aligned} \eta(\theta; f, g) &\geq \deg_{\mathbf{w}}^g \Psi + a(\deg_{\mathbf{w}} df \wedge dg - p\delta - q\delta) \\ &\geq (ap+b)q\delta + a(\deg_{\mathbf{w}} df \wedge dg - p\delta - q\delta). \end{aligned} \tag{2.4}$$

First, assume that (i) is satisfied. Then, we have $(p,q) = (l, 2l+1)$. Since $\deg_{\mathbf{w}} df \wedge dg > \deg_{\mathbf{w}} g = q\delta$ by assumption, and $a \geq 1$ and $b \geq 0$, it follows that

$$\eta(\theta; f, g) > (ap)q\delta + a(q-p-q)\delta = ap(q-1)\delta \geq 2l^2\delta$$

by (2.4). By (2.3), we have $\eta_1 = (4l + 3/2)\delta$ and $\eta_2 = (4l+1)\delta$. Since $l \geq 3$ by assumption, we know that $\eta_i < 2l^2\delta$ for $i = 1, 2$. Therefore, we conclude that $\eta(\theta; f, g) > \eta_i$ for $i = 1, 2$.

Next, assume that (ii) is satisfied. Then, we have $\deg_{\mathbf{w}} df \wedge dg > 2\deg_{\mathbf{w}} f = 2p\delta$. Hence, we get

$$\eta(\theta; f, g) > (ap+b)q\delta + a\big(2p-p-q\big)\delta = \Big(a\big(p + (p-1)q\big) + bq\Big)\delta =: \alpha$$

by (2.4). We show that $\alpha \geq (p+2q)\delta$. If $p \geq 3$, then this is clear, since $a \geq 1$ and $b \geq 0$. Assume that $p = 2$. Then, we have $2a + b = \deg_z \Psi \geq d \geq 3$ with $0 \leq b \leq 1$. Hence, we get $a \geq 2$ or $(a, b) = (1, 1)$. Thus, we know that $\alpha \geq (p+2q)\delta$. Since $p < q$, we see from (2.3) that $\eta_i < (p+2q)\delta$ for $i = 1, 2$. Therefore, we conclude that $\eta(\theta; f, g) > \eta_i$ for $i = 1, 2$.

Finally, assume that (iii) is satisfied. By (3), we have $df \wedge dg \neq 0$. Since $\mathbf{w}$ is an element of $(\Gamma_{>0})^3$, it follows that $\deg_{\mathbf{w}} df \wedge dg > 0$. Hence, (2.4) gives that

$$\begin{aligned} \eta(\theta; f, g) &> \deg_{\mathbf{w}}^g \Psi - a(p+q)\delta \\ &\geq (ap+b)q\delta - a(p+q)\delta = \Big(a\big((p-1)q - p\big) + bq\Big)\delta =: \beta. \end{aligned} \tag{2.5}$$

Note that $\beta > a(p-2)q\delta$, since $a \geq 1$, $b \geq 0$ and $p < q$. First, assume that $p \geq 3$. We show that $a(p-2) \geq 3$. Since $a \geq 1$, this is clear if $p \geq 5$. If $p = 4$, then we have $a \geq 2$, since $4a+b = \deg_z \Psi \geq d \geq 9$ with $0 \leq b \leq 3$. Hence, we get $a(p-2) \geq 4$. If $p = 3$, then we have $a \geq 3$, since $3a+b = \deg_z \Psi \geq d \geq 9$ with $0 \leq b \leq 2$. Hence, we get $a(p-2) \geq 3$. Thus, we know that $\beta > 3q\delta$. Because $\eta_i < 3q\delta$ for $i = 1, 2$, we conclude that $\eta(\theta; f, g) > \eta_i$ for $i = 1, 2$. Next, assume that $p = 2$. Then, we have $\eta_1 = ((3/2)q+2)\delta$ and $\eta_2 = (q+4)\delta$

by (2.3). Since $q \geq p+1 = 3$, it follows that $\eta_i \leq (5q-8)\delta$ for $i = 1, 2$. First, consider the case where $\deg_z \Psi \neq 10, 12$. Since $2a + b = \deg_z \Psi \geq d \geq 9$ with $0 \leq b \leq 1$, we know that $a \geq 7$, or $4 \leq a \leq 6$ and $b = 1$. If $a \geq 7$, then we have

$$\beta \geq 7(q-2)\delta = \big(5q + (2q-14)\big)\delta \geq (5q-8)\delta,$$

since $p = 2$ and $q \geq 3$. If $4 \leq a \leq 6$ and $b = 1$, then we have

$$\beta \geq (4(q-2) + q)\delta = (5q-8)\delta.$$

Therefore, we conclude that $\eta(\theta; f, g) > \eta_i$ for $i = 1, 2$. Next, consider the case where $\deg_z \Psi = 10$ or $\deg_z \Psi = 12$. Since $d \neq 10, 12$ by assumption, $\deg_z \Psi$ is not equal to $d = \deg_z \theta$. Hence, we have $\deg_z \theta < \deg_z f\Phi$ by (2.1). Since θ is an element of $k[z]$, we get

$$\deg_{\mathbf{w}_g} \theta = (\deg_z \theta) \deg_{\mathbf{w}} g < (\deg_z f\Phi) \deg_{\mathbf{w}} g \leq \deg_{\mathbf{w}_g} f\Phi.$$

Thus, we obtain

$$\begin{aligned} \deg^g_{\mathbf{w}} \Psi &= \deg_{\mathbf{w}_g}(\theta + f\Phi) = \deg_{\mathbf{w}_g} f\Phi = \deg_{\mathbf{w}} f + \deg_{\mathbf{w}_g} \Phi \\ &\geq \deg_{\mathbf{w}} f + (\deg_z \Phi) \deg_{\mathbf{w}} g = \big(2 + (\deg_z \Phi) q\big)\delta. \end{aligned}$$

Since $\deg_z \theta < \deg_z f\Phi$, we have $\deg_z \Phi = \deg_z f\Phi = \deg_z \Psi$ by (2.1). Because $\deg_z \Psi = 2a + b \geq 2a$, it follows that $\deg^g_{\mathbf{w}} \Psi \geq (2 + 2aq)\delta$ by the preceding inequality. Hence, the first inequality of (2.5) gives that

$$\eta(\theta; f, g) > \deg^g_{\mathbf{w}} \Psi - a(2+q)\delta \geq (2 + 2aq)\delta - a(2+q)\delta = \big(a(q-2) + 2\big)\delta.$$

Since $2a + b = \deg_z \Psi \geq 10$ with $0 \leq b \leq 1$, we have $a \geq 5$. Hence, we know that $a(q-2) + 2 \geq 5(q-2) + 2 = 5q - 8$, since $q \geq 3$. Therefore, we get $\eta(\theta; f, g) > \eta_i$ for $i = 1, 2$. □

The following criterion for wildness will be used in Section 2.

Lemma 2.4. $F = (f_1, f_2, f_3) \in \mathrm{Aut}(k[\mathbf{x}]/k)$ *is wild if the following conditions hold*:
(a) $f_1^{\mathbf{w}}$, $f_2^{\mathbf{w}}$ *and* $f_3^{\mathbf{w}}$ *are algebraically dependent over* k.
(b) $f_1^{\mathbf{w}}$ *and* $f_2^{\mathbf{w}}$ *do not belong to* $k[f_2, f_3]^{\mathbf{w}}$ *and* $k[f_3^{\mathbf{w}}]$, *respectively.*
(c) $\deg_{\mathbf{w}} f_1 \geq \deg_{\mathbf{w}} f_2 + \deg_{\mathbf{w}} f_3$.
(d) $\deg_{\mathbf{w}} f_2 > \deg_{\mathbf{w}} f_3$.
(e) $2\deg_{\mathbf{w}} f_1 \neq 3\deg_{\mathbf{w}} f_2$.

Proof. By Lemma 1.1, (a) implies that $\deg_{\mathbf{w}} \phi > |\mathbf{w}|$. Hence, it suffices to check that ϕ admits no elementary reduction for the weight $\mathbf{w}$, and no Shestakov-Umirbaev reduction for the weight $\mathbf{w}$ by virtue of Theorem 1.3. Suppose to the contrary that ϕ admits a Shestakov-Umirbaev reduction for the weight $\mathbf{w}$. By definition, there exist σ and G such that (F_σ, G_σ) satisfies the Shestakov-Umirbaev condition for the weight $\mathbf{w}$. Then, (F_σ, G_σ) has the properties listed before Lemma 1.1. By (P7), we know that $f_{\sigma(1)} > f_{\sigma(2)}$ and $f_{\sigma(1)} \geq f_{\sigma(3)}$. Since $\deg_{\mathbf{w}} f_1 > \deg_{\mathbf{w}} f_2 > \deg_{\mathbf{w}} f_3$ by (c) and (d), it follows that $\sigma(1) = 1$. Hence, we have $\deg_{\mathbf{w}} g_1 = s\delta$ and $\deg_{\mathbf{w}} g_{\sigma(2)} = 2\delta$ for some $s \geq 3$ by (P1), and $\deg_{\mathbf{w}} g_1 \geq \deg_{\mathbf{w}} f_1$ and $\deg_{\mathbf{w}} g_{\sigma(2)} = \deg_{\mathbf{w}} f_{\sigma(2)}$ by (SU2). Thus, we get

$$\deg_{\mathbf{w}} f_{\sigma(3)} > (s-2)\delta = \deg_{\mathbf{w}} g_1 - \deg_{\mathbf{w}} g_{\sigma(2)} \geq \deg_{\mathbf{w}} f_1 - \deg_{\mathbf{w}} f_{\sigma(2)}$$

by (P2). This contradicts (c). Therefore, ϕ admits no Shestakov-Umirbaev reduction for the weight $\mathbf{w}$.

Next, we show that ϕ admits no elementary reduction for the weight $\mathbf{w}$. Since $f_1^{\mathbf{w}}$ does not belong to $k[f_2, f_3]^{\mathbf{w}}$ by (b), we check that $f_2^{\mathbf{w}}$ and $f_3^{\mathbf{w}}$ do not belong to $k[f_1, f_3]^{\mathbf{w}}$ and $k[f_1, f_2]^{\mathbf{w}}$, respectively.

First, suppose to the contrary that $f_2^{\mathbf{w}}$ belongs to $k[f_1, f_3]^{\mathbf{w}}$. Then, there exists $h \in k[f_1, f_3]$ such that $h^{\mathbf{w}} = f_2^{\mathbf{w}}$. We show that $\deg_{\mathbf{w}} f_2 = \deg_{\mathbf{w}} h$ is greater than $\deg_{\mathbf{w}} f_1 - \deg_{\mathbf{w}} f_3$ by applying Lemma 2.2 (iii) with $f = f_3$, $g = f_1$ and $\phi = h$. Then, we get a contradiction to (c). Since $f_2^{\mathbf{w}}$ does not belong to $k[f_3^{\mathbf{w}}]$ by (b), we know that h belongs to $k[f_1, f_3] \setminus k[f_3]$. Since $\deg_{\mathbf{w}} h = \deg_{\mathbf{w}} f_2$ is less than $\deg_{\mathbf{w}} f_1$ by (c), this implies that $\deg_{\mathbf{w}} h < \deg_{\mathbf{w}}^{S_2} h$, where $S_2 := \{f_1, f_3\}$. By (c), we have $\deg_{\mathbf{w}} f_1 > \deg_{\mathbf{w}} f_3$, and $\deg_{\mathbf{w}} f_1 > \deg_{\mathbf{w}} f_2 = \deg_{\mathbf{w}} h$. By (b), $f_1^{\mathbf{w}}$ does not belong to $k[f_2^{\mathbf{w}}, f_3^{\mathbf{w}}]$, and hence does not belong to $k[f_3^{\mathbf{w}}]$. Thus, we conclude from Lemma 2.2 (iii) that $\deg_{\mathbf{w}} h > \deg_{\mathbf{w}} f_1 - \deg_{\mathbf{w}} f_3$. This proves that $f_2^{\mathbf{w}}$ does not belong to $k[f_1, f_3]^{\mathbf{w}}$.

Next, suppose to the contrary that $f_3^{\mathbf{w}}$ belongs to $k[f_1, f_2]^{\mathbf{w}}$. Then, there exists $h \in k[f_1, f_2]$ such that $h^{\mathbf{w}} = f_3^{\mathbf{w}}$. We show that $(f_1^{\mathbf{w}})^2 \approx (f_2^{\mathbf{w}})^3$ by applying Lemma 2.2 (i) and the last part of Lemma 2.2 (iii) with $f = f_2$, $g = f_1$ and $\phi = h$. Then, we get a contradiction to (e). By (c) and (d), we have $\deg_{\mathbf{w}} h = \deg_{\mathbf{w}} f_3 < \deg_{\mathbf{w}} f_i$ for $i = 1, 2$. Hence, we get $\deg_{\mathbf{w}} h < \deg_{\mathbf{w}}^{S_3} h$, where $S_3 := \{f_1, f_2\}$. By (c) and (d), we have $\deg_{\mathbf{w}} f_1 > \deg_{\mathbf{w}} f_2$ and $\deg_{\mathbf{w}} f_2 > \deg_{\mathbf{w}} f_3 = \deg_{\mathbf{w}} h$. By (b), $f_1^{\mathbf{w}}$ does not belong to $k[f_2^{\mathbf{w}}, f_3^{\mathbf{w}}]$, and hence does not belong to $k[f_2^{\mathbf{w}}]$. Thus, we conclude from Lemma 2.2 (i) and the last part of Lemma 2.2 (iii) that $(f_1^{\mathbf{w}})^2 \approx (f_2^{\mathbf{w}})^3$. This proves that $f_3^{\mathbf{w}}$ does not belong to $k[f_1, f_2]^{\mathbf{w}}$. $\square$

CHAPTER 6

Totally wild coordinates

1. Main result

In what follows, we always assume that $n = 3$ unless otherwise stated. The purpose of this chapter is to give coordinates of $k[\mathbf{x}]$ over k some of which are totally wild, and others are quasi-totally wild, but not totally wild.

Let $\theta(z)$ be an element of $k[z]$ with $d := \deg_z \theta(z) \geq 1$. Define $D_\theta \in \mathrm{Der}_k\, k[\mathbf{x}]$ by

$$D_\theta(x_1) = -\theta'(x_2), \quad D_\theta(x_2) = x_3, \quad D_\theta(x_3) = 0.$$

Then, D_θ is locally nilpotent, since D_θ is triangular if x_1 and x_3 are interchanged. Since $f_\theta := x_1x_3 + \theta(x_2)$ belongs to $\ker D_\theta$, it follows that $f_\theta D_\theta$ is locally nilpotent. Set $\sigma_\theta = \exp f_\theta D_\theta$ and $y_i = \sigma_\theta(x_i)$ for $i = 1, 2, 3$. Then, we consider the tame intersection

$$G_\theta := \mathrm{Aut}(k[\mathbf{x}]/k[y_1]) \cap \mathrm{T}(k, \mathbf{x}).$$

Let us describe y_1 concretely. Since $D_\theta(f_\theta) = D_\theta(x_3) = 0$, we have $\sigma_\theta(f_\theta) = f_\theta$ and $y_3 = x_3$, and so

$$y_1x_3 + \theta(y_2) = \sigma_\theta(x_1x_3 + \theta(x_2)) = \sigma_\theta(f_\theta) = f_\theta = x_1x_3 + \theta(x_2).$$

Hence, we get

$$y_1 = x_1 + \frac{\theta(x_2) - \theta(y_2)}{x_3}. \tag{1.1}$$

Note that $y_2 = x_2 + f_\theta x_3$, since $(f_\theta D_\theta)(x_2) = f_\theta x_3$ and $(f_\theta D_\theta)^2(x_2) = 0$. Therefore, (1.1) gives that

$$y_1 = x_1 - \frac{\theta(x_2 + f_\theta x_3) - \theta(x_2)}{x_3} = x_1 - \sum_{i=1}^{d} \frac{1}{i!}\theta^{(i)}(x_2) f_\theta^i x_3^{i-1}, \tag{1.2}$$

where $\theta^{(i)}(z)$ denotes the i-th order derivative of $\theta(z)$.

Now, let c and c' be the coefficients of z^d and z^{d-1} in $\theta(z)$, respectively. Put $\kappa = -c'/(cd)$ and write

$$\theta(z) = \sum_{i=0}^{d} u_i (z - \kappa)^i,$$

where $u_i \in k$ for each i. Then, we have $u_d = c$, $u_{d-1} = 0$ and $u_0 = \theta(\kappa)$. Let $e \in \mathbf{N}$ be the greatest common divisor of $U := \{2i-1 \mid i = 1, \ldots, d \text{ with } u_i \neq 0\}$, i.e., the positive generator of the ideal (U) of $\mathbf{Z}$. Then, we define

$$T_\theta = \{\zeta \in k^\times \mid \zeta^e = 1\}.$$

For each $\zeta \in T_\theta$, we define an element ϕ_ζ of $J(k;x_3,x_2,x_1)$ by

$$\begin{aligned}\phi_\zeta(x_1) &= x_1 + g_\zeta\\ \phi_\zeta(x_2) &= \zeta^2(x_2-\kappa) + \zeta(\zeta-1)\theta(\kappa)x_3 + \kappa\\ \phi_\zeta(x_3) &= \zeta x_3,\end{aligned}$$

where

$$g_\zeta := \frac{\zeta\theta(x_2) - \phi_\zeta(\theta(x_2)) + (1-\zeta)\theta(\kappa)}{\zeta x_3}.$$

Here, we note that g_ζ belongs to $k[\mathbf{x}]$ if and only if ζ belongs to T_θ for $\zeta \in k^\times$. To see this, observe that the numerator of g_ζ is congruent to

$$\zeta\sum_{i=0}^{d} u_i(x_2-\kappa)^i - \sum_{i=0}^{d} u_i\bigl(\zeta^2(x_2-\kappa)\bigr)^i + (1-\zeta)\theta(\kappa) = \sum_{i=1}^{d} u_i\zeta(1-\zeta^{2i-1})(x_2-\kappa)^i$$

modulo $x_3k[\mathbf{x}]$, since $\theta(\kappa) = u_0$. Then, the right-hand side of this equality belongs to $x_3k[\mathbf{x}]$ if and only if it is equal to zero. This condition is satisfied if and only if $\zeta^j = 1$ for each $j \in U$, and hence if and only if ζ belongs to T_θ.

The following is the main result.

Theorem 1.1. *In the notation above, ϕ_ζ belongs to G_θ for each $\zeta \in T_\theta$, and*

$$\iota : T_\theta \ni \zeta \mapsto \phi_\zeta \in G_\theta$$

is an injective homomorphism of groups. If $d \geq 9$ and $d \neq 10, 12$, then ι is surjective.

Theorem 1.1 immediately implies the following corollary.

Corollary 1.2. *Assume that $d \geq 9$ and $d \neq 10, 12$. Then, the following assertions hold*:
(i) *y_1 is a quasi-totally wild coordinate of $k[\mathbf{x}]$ over k.*
(ii) *y_1 is a totally wild coordinate of $k[\mathbf{x}]$ over k if and only if $T_\theta = \{1\}$.*

Recall that k is said to be *real* if -1 is not a sum of squares in k. We remark that the roots of unity in a real field are only 1 and -1. Actually, real fields are ordered fields (cf. [**18**, Chapter XI, Section 2]). Hence, if $\zeta^2 \neq 1$, then we have $\zeta^2 > 1$ or $0 \leq \zeta^2 < 1$. If $\zeta^e = 1$ for some $e \geq 1$, then it follows that $1 = (\zeta^2)^e > 1^e = 1$ or $1 = (\zeta^2)^e < 1^e = 1$, a contradiction.

By the following proposition, we see that there exist a number of totally wild coordinates.

Proposition 1.3. *If one of the following conditions is satisfied, then we have $T_\theta = \{1\}$*:
(1) *k is a real field.*
(2) *$u_{d-i} \neq 0$ for some $i \geq 2$ such that* $\gcd(i, 2d-1) = 1$.

Proof. Since u_d is nonzero, e must be a divisor of $2d-1$. Hence, e is an odd number. This implies that $T_\theta = \{1\}$ if k is a real field.

If $u_{d-i} \neq 0$, then e divides $2(d-i)-1$. Hence, e divides

$$\gcd(2(d-i)-1, 2d-1) = \gcd(2i, 2d-1) = \gcd(i, 2d-1) = 1.$$

Thus, we have $e = 1$. Therefore, we get $T_\theta = \{1\}$. $\square$

We remark that σ_{z^2} is equal to Nagata's automorphism defined in (0.2). Clearly, $y_3 = x_3$ is a tame coordinate, while y_1 and y_2 are wild coordinates due to Umirbaev-Yu [**29**]. Observe that $y_2 = x_1x_3^2 + (x_2 + x_2^2x_3)$ is killed by $D \in \mathrm{Der}_k\, k[\mathbf{x}]$ defined by

$$D(x_1) = 1 + 2x_2x_3, \quad D(x_2) = -x_3^2, \quad D(x_3) = 0. \tag{1.3}$$

Since D is triangular if x_1 and x_3 are interchanged, we know that $\exp D$ is tame. Hence, y_2 is not exponentially wild. Thus, a wild coordinate of $k[\mathbf{x}]$ is not always exponentially wild. Since $\theta(z) = z^2$, we have $e = 2d - 1 = 3$. Hence, we get $T_{z^2} = \{\zeta \in k \mid \zeta^3 = 1\}$. Therefore, y_1 is not totally wild if k contains a primitive third root of unity. The author believes that y_1 is quasi-totally wild, but it remains open.

In the rest of this section, we prove the first part of Theorem 1.1.

Lemma 1.4. *For each $\zeta \in T_\theta$, we have*

$$\phi_\zeta(y_1) = y_1, \quad \phi_\zeta(y_2 - \kappa) = \zeta^2(y_2 - \kappa), \quad \phi_\zeta(y_3) = \zeta y_3.$$

If furthermore $\theta(\kappa) = 0$, then we have

$$\phi_\zeta(x_1) = x_1, \quad \phi_\zeta(x_2 - \kappa) = \zeta^2(x_2 - \kappa), \quad \phi_\zeta(x_3) = \zeta x_3,$$

and so $\sigma_\theta \circ \phi_\zeta = \phi_\zeta \circ \sigma_\theta$.

Proof. Write $\phi = \phi_\zeta$ for simplicity. Since $y_3 = x_3$, we have $\phi(y_3) = \zeta y_3$ by the definition of ϕ_ζ. A direct computation shows that

$$\begin{aligned}\phi(f_\theta) &= \phi\big(x_1x_3 + \theta(x_2)\big) = (x_1 + g_\zeta)(\zeta x_3) + \theta\big(\phi(x_2)\big)\\ &= \zeta\big(x_1x_3 + \theta(x_2)\big) - \zeta\theta(x_2) + g_\zeta(\zeta x_3) + \theta\big(\phi(x_2)\big) = \zeta f_\theta + (1 - \zeta)\theta(\kappa).\end{aligned}$$

Since $y_2 = x_2 + f_\theta x_3$, it follows that

$$\begin{aligned}\phi(y_2 - \kappa) &= \phi(x_2 - \kappa) + \phi(f_\theta)\phi(x_3)\\ &= \zeta^2(x_2 - \kappa) + \zeta(\zeta - 1)\theta(\kappa)x_3 + \big(\zeta f_\theta + (1 - \zeta)\theta(\kappa)\big)(\zeta x_3)\\ &= \zeta^2(x_2 - \kappa + f_\theta x_3) = \zeta^2(y_2 - \kappa).\end{aligned}$$

Hence, we have

$$\phi\big(\theta(y_2)\big) = \sum_{i=0}^{d} u_i\phi(y_2 - \kappa)^i = \zeta\sum_{i=1}^{d} u_i\zeta^{2i-1}(y_2 - \kappa)^i + u_0. \tag{1.4}$$

Since ζ is an element of T_θ, we have $\zeta^{2i-1} = 1$ for each $i \geq 1$ with $u_i \neq 0$. Thus, the right-hand side of (1.4) is equal to

$$\zeta\sum_{i=1}^{d} u_i(y_2 - \kappa)^i + u_0 = \zeta\theta(y_2) + (1 - \zeta)u_0. \tag{1.5}$$

Therefore, it follows from (1.1) that

$$\begin{aligned}\phi(y_1) &= \phi\left(x_1 + \frac{\theta(x_2) - \theta(y_2)}{x_3}\right)\\ &= x_1 + g_\zeta + \frac{\phi\big(\theta(x_2)\big) - \big(\zeta\theta(y_2) + (1 - \zeta)u_0\big)}{\zeta x_3}\\ &= x_1 + \frac{\zeta\theta(x_2) - \zeta\theta(y_2)}{\zeta x_3} = y_1.\end{aligned}$$

Next, assume that $\theta(\kappa)=0$. Then, we have $\phi(x_2-\kappa)=\zeta^2(x_2-\kappa)$. Hence, we obtain $\phi(\theta(x_2))=\zeta\theta(x_2)+(1-\zeta)\theta(\kappa)$ from (1.4) and (1.5) with y_2 replaced by x_2. This implies that $g_\zeta=0$. Thus, we get $\phi(x_1)=x_1$. By definition, we have $\phi(x_3)=\zeta x_3$. Since $\sigma_\theta(x_i)=y_i$ for $i=1,2,3$, it is easy to see that $\sigma_\theta\circ\phi=\phi\circ\sigma_\theta$. □

Since ϕ_ζ is tame, and $\phi_\zeta(y_1)=y_1$ by Lemma 1.4, we know that ϕ_ζ belongs to G_θ for each $\zeta\in T_\theta$. By Lemma 1.4, we have

$$(\phi_\alpha\circ\phi_\beta)(y_1)=y_1=\phi_{\alpha\beta}(y_1)$$
$$(\phi_\alpha\circ\phi_\beta)(y_2-\kappa)=\alpha^2\beta^2(y_2-\kappa)=\phi_{\alpha\beta}(y_2-\kappa)$$
$$(\phi_\alpha\circ\phi_\beta)(y_3)=\alpha\beta y_3=\phi_{\alpha\beta}(y_3)$$

for each $\alpha,\beta\in T_\theta$. Hence, ι is a homomorphism of groups. If $\phi_\zeta=\mathrm{id}_{k[\mathbf{x}]}$ for $\zeta\in T_\theta$, then we have $\zeta x_3=\phi_\zeta(x_3)=x_3$, and hence $\zeta=1$. Therefore, ι is injective.

Sections 2, 3 and 4 are devoted to proving the following theorem. Let Λ be the totally ordered additive group $\mathbf{Z}^2$ equipped with the lexicographic order such that $\mathbf{e}_1>\mathbf{e}_2$. We consider the element $\mathbf{v}:=(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_1)$ of Λ^3.

Theorem 1.5. (i) *Let Γ be a totally ordered additive group, and $\mathbf{w}\in(\Gamma_{>0})^3$. If $d\geq 9$ and $d\neq 10,12$, then we have $\deg_{\mathbf{w}}\phi(y_2)>\deg_{\mathbf{w}}\phi(x_3)$ for each $\phi\in G_\theta$.*
(ii) *Let $\mathbf{v}$ be as above, and assume that $d\geq 9$. If $\deg_{\mathbf{v}}\phi(y_2)>\deg_{\mathbf{v}}\phi(x_3)$ for $\phi\in G_\theta$, then there exists $\zeta\in T_\theta$ such that $\phi=\phi_\zeta$.*

By this theorem, it follows that ι is surjective if $d\geq 9$ and $d\neq 10,12$. Thus, the proof of Theorem 1.1 is completed.

2. Proof (I)

Let ϕ be an element of $\mathrm{Aut}(k[\mathbf{x}]/k[y_1])$. First, we investigate when ϕ is wild in general. Set $z_i=\phi(x_i)$ for $i=1,2,3$. Then, we have

$$\begin{aligned} z_1&=\big(\phi(f_\theta)-\theta(z_2)\big)z_3^{-1}\\ z_2&=\phi(y_2-f_\theta x_3)\\ z_3&=\phi(y_3),\end{aligned}\tag{2.1}$$

since $z_1z_3+\theta(z_2)=\phi(f_\theta)$, $x_2=y_2-f_\theta x_3$, and $x_3=y_3$. Let $\mathbf{w}=(w_1,w_2,w_3)$ be an element of $(\Gamma_{>0})^3$. Then, there exist

$$\gamma_0^{\mathbf{w}}:=\min\{\deg_{\mathbf{w}}\big(\phi(f_\theta)+h_0\big)\mid h_0\in k[z_3]\}$$
$$\gamma_1^{\mathbf{w}}:=\min\{\deg_{\mathbf{w}}\big(\theta(z_2)+h_1z_3\big)\mid h_1\in k[z_2,z_3]\}$$
$$\gamma_2^{\mathbf{w}}:=\min\{\deg_{\mathbf{w}}(z_2+h_2)\mid h_2\in k[z_3]\}$$

by the well-orderedness of $\{\deg_{\mathbf{w}}h\mid h\in k[\mathbf{x}]\setminus\{0\}\}$.

Take $h_0,h_2\in k[z_3]$ and $h_1\in k[z_2,z_3]$ such that

$$\gamma_0^{\mathbf{w}}=\deg_{\mathbf{w}}\big(\phi(f_\theta)+h_0\big),\quad \gamma_1^{\mathbf{w}}=\deg_{\mathbf{w}}\big(\theta(z_2)+h_1z_3\big),\quad \gamma_2^{\mathbf{w}}=\deg_{\mathbf{w}}(z_2+h_2).$$

Then, $\phi(f_\theta)+h_0$ and z_2+h_2 do not belong to k, for otherwise $\phi(f_\theta)$ or z_2 belongs to $k[z_3]$, and so f_θ or x_2 belongs to $k[x_3]$, a contradiction. Similarly, $\theta(z_2)+h_1z_3$ does not belong to k, since z_2 and z_3 are algebraically independent over k. Hence, we have $\gamma_i^{\mathbf{w}}>0$ for $i=0,1,2$ by the choice of $\mathbf{w}$.

For this reason, the $\mathbf{w}$-degrees of $\phi(f_\theta)+h_0$ and z_2+h_2 are not changed if we subtract the constant terms from h_0 and h_2. Therefore, we may assume that h_0 and h_2 are elements of $z_3k[z_3]$. We remark that $\big(\phi(f_\theta)+h_0\big)^{\mathbf{w}}$ and $(z_2+h_2)^{\mathbf{w}}$ do not belong to $k[z_3]^{\mathbf{w}}=k[z_3^{\mathbf{w}}]$ by the minimality of $\gamma_0^{\mathbf{w}}$ and $\gamma_2^{\mathbf{w}}$.

Set $\gamma_3^{\mathbf{w}}=\deg_{\mathbf{w}} z_3>0$. Then, we have the following lemma.

Lemma 2.1. *If $\deg_{\mathbf{w}}\phi(y_2)<\gamma_0^{\mathbf{w}}+\gamma_3^{\mathbf{w}}$, then we have $\gamma_i^{\mathbf{w}}<\gamma_2^{\mathbf{w}}$ for $i=0,3$.*

Proof. Since $\gamma_i^{\mathbf{w}}>0$ for $i=0,3$, it suffices to show that $\gamma_0^{\mathbf{w}}+\gamma_3^{\mathbf{w}}\leq\gamma_2^{\mathbf{w}}$. Since $h_2z_3^{-1}$ belongs to $k[z_3]$ by the choice of h_2, we have $\deg_{\mathbf{w}}(\phi(f_\theta)-h_2z_3^{-1})\geq\gamma_0^{\mathbf{w}}$ by the minimality of $\gamma_0^{\mathbf{w}}$. Hence, we get

$$\deg_{\mathbf{w}}(\phi(f_\theta)-h_2z_3^{-1})z_3\geq\gamma_0^{\mathbf{w}}+\gamma_3^{\mathbf{w}}>\deg_{\mathbf{w}}\phi(y_2)\tag{2.2}$$

by assumption. Since $z_2=\phi(y_2)-\phi(f_\theta)z_3$ by (2.1), we have

$$\gamma_2^{\mathbf{w}}=\deg_{\mathbf{w}}(z_2+h_2)=\deg_{\mathbf{w}}\big(\phi(y_2)-(\phi(f_\theta)-h_2z_3^{-1})z_3\big).$$

Thus, we see from (2.2) that $\gamma_2^{\mathbf{w}}$ is equal to the left-hand side of (2.2), and is at least $\gamma_0^{\mathbf{w}}+\gamma_3^{\mathbf{w}}$. Therefore, we conclude that $\gamma_i^{\mathbf{w}}<\gamma_2^{\mathbf{w}}$ for $i=0,3$. □

We define $\psi\in\mathrm{Aut}(k[\mathbf{x}]/k)$ by

$$\psi(x_1)=z_1+h_0z_3^{-1}-h_1,\quad \psi(x_2)=z_2+h_2,\quad \psi(x_3)=z_3.$$

Then, $\phi^{-1}\circ\psi$ belongs to $J(k[x_3];x_2,x_1)$. Hence, ϕ is tame if and only if so is ψ. We note that $\deg_{\mathbf{w}}\psi(x_i)=\gamma_i^{\mathbf{w}}$ for $i=2,3$.

Lemma 2.2. *Assume that $\gamma_0^{\mathbf{w}}<\gamma_1^{\mathbf{w}}$, $\gamma_2^{\mathbf{w}}>\gamma_3^{\mathbf{w}}$ and*

$$2\gamma_1^{\mathbf{w}}>3\gamma_2^{\mathbf{w}}+2\gamma_3^{\mathbf{w}},\quad \gamma_1^{\mathbf{w}}\geq\gamma_2^{\mathbf{w}}+2\gamma_3^{\mathbf{w}}\tag{2.3}$$

for some $\mathbf{w}\in(\Gamma_{>0})^3$. Then, ϕ is wild.

Proof. It suffices to prove that ψ is wild. We check that ψ satisfies the conditions (a) through (e) of Proposition 2.4. Since $\deg_{\mathbf{w}}\psi(x_i)=\gamma_i^{\mathbf{w}}$ for $i=2,3$, and $\gamma_2^{\mathbf{w}}>\gamma_3^{\mathbf{w}}$ by assumption, we see that ψ satisfies (d). By (2.1), we have

$$\psi(x_1)=\big(\phi(f_\theta)-\theta(z_2)\big)z_3^{-1}+h_0z_3^{-1}-h_1=\big((\phi(f_\theta)+h_0)-(\theta(z_2)+h_1z_3)\big)z_3^{-1}.$$

Since $\deg_{\mathbf{w}}\big(\phi(f_\theta)+h_0\big)=\gamma_0^{\mathbf{w}}<\gamma_1^{\mathbf{w}}=\deg_{\mathbf{w}}\big(\theta(z_2)+h_1z_3\big)$ by assumption, it follows that

$$\psi(x_1)^{\mathbf{w}}=-\big(\theta(z_2)+h_1z_3\big)^{\mathbf{w}}(z_3^{\mathbf{w}})^{-1}.\tag{2.4}$$

Hence, $\psi(x_1)^{\mathbf{w}}$ belongs to the field of fractions of $k[z_2,z_3]^{\mathbf{w}}$. Since $\psi(x_2)$ and $\psi(x_3)$ belong to $k[z_2,z_3]$, we know that $\psi(x_2)^{\mathbf{w}}$ and $\psi(x_3)^{\mathbf{w}}$ also belong to $k[z_2,z_3]^{\mathbf{w}}$. Thus, $\psi(x_1)^{\mathbf{w}}$, $\psi(x_2)^{\mathbf{w}}$ and $\psi(x_3)^{\mathbf{w}}$ are algebraically dependent over k, since the field of fractions of $k[z_2,z_3]^{\mathbf{w}}$ has transcendence degree at most two over k. Therefore, ψ satisfies (a). Since $\deg_{\mathbf{w}}\psi(x_1)=\gamma_1^{\mathbf{w}}-\gamma_3^{\mathbf{w}}$ by (2.4), we have

$$2\deg_{\mathbf{w}}\psi(x_1)=2\gamma_1^{\mathbf{w}}-2\gamma_3^{\mathbf{w}}>3\gamma_2^{\mathbf{w}}=3\deg_{\mathbf{w}}\psi(x_2)$$

by the first part of (2.3). This proves (e). Similarly, we have

$$\deg_{\mathbf{w}}\psi(x_1)=\gamma_1^{\mathbf{w}}-\gamma_3^{\mathbf{w}}\geq\gamma_2^{\mathbf{w}}+\gamma_3^{\mathbf{w}}=\deg_{\mathbf{w}}\psi(x_2)+\deg_{\mathbf{w}}\psi(x_3)$$

by the second part of (2.3). This proves (c). Since $\psi(x_2)^{\mathbf{w}}=(z_2+h_2)^{\mathbf{w}}$ does not belong to $k[\psi(x_3)^{\mathbf{w}}]=k[z_3^{\mathbf{w}}]$ as mentioned, ψ satisfies the second

part of (b). We prove the first part of (b) by contradiction. Suppose that $\psi(x_1)^{\mathbf{w}}$ belongs to $k[\psi(x_2),\psi(x_3)]^{\mathbf{w}}$. Then, there exists $h\in k[\psi(x_2),\psi(x_3)]$ such that $h^{\mathbf{w}}=\psi(x_1)^{\mathbf{w}}$. By (2.4), it follows that

$$(hz_3)^{\mathbf{w}}+(\theta(z_2)+h_1z_3)^{\mathbf{w}}=h^{\mathbf{w}}z_3^{\mathbf{w}}-\psi(x_1)^{\mathbf{w}}z_3^{\mathbf{w}}=0.$$

Hence, the $\mathbf{w}$-degree of

$$\theta(z_2)+(h_1+h)z_3=hz_3+\bigl(\theta(z_2)+h_1z_3\bigr)$$

is less than $\deg_{\mathbf{w}}(\theta(z_2)+h_1z_3)=\gamma_1^{\mathbf{w}}$. By the minimality of $\gamma_1^{\mathbf{w}}$, this implies that h_1+h does not belong to $k[z_2,z_3]$. However, h_1 is an element of $k[z_2,z_3]$, and h is an element of $k[\psi(x_2),\psi(x_3)]=k[z_2+h_2,z_3]=k[z_2,z_3]$. Hence, h_1+h belongs to $k[z_2,z_3]$, a contradiction. This proves the first part of (b). Thus, ψ satisfies (a) through (e) of Proposition 2.4. Therefore, we conclude that ψ is wild. Consequently, ϕ is wild. $\square$

Since h_2 belongs to $z_3k[x_3]$, we see that $\theta\bigl(\psi(x_2)\bigr)=\theta(z_2+h_2)$ has the form $\theta(z_2)+hz_3$ for some $h\in k[z_2,z_3]$. Similarly, $\theta(z_2)=\theta\bigl(\psi(x_2)-h_2\bigr)$ has the form $\theta\bigl(\psi(x_2)\bigr)+h'z_3$ for some $h'\in k[\psi(x_2),z_3]$. Since $k[\psi(x_2),z_3]=k[z_2,z_3]$, it follows that

$$\{\theta\bigl(\psi(x_2)\bigr)+hz_3\mid h\in k[\psi(x_2),z_3]\}=\{\theta(z_2)+hz_3\mid h\in k[z_2,z_3]\}.$$

Hence, we have

$$\gamma_1^{\mathbf{w}}=\min\{\deg_{\mathbf{w}}\bigl(\theta(\psi(x_2))+hz_3\bigr)\mid h\in k[\psi(x_2),z_3]\}. \tag{2.5}$$

In the notation of Lemma 2.3, we may write $\gamma_1^{\mathbf{w}}=\eta(\theta;z_3,\psi(x_2))$. We note that the conditions (2) and (3) before Lemma 2.3 are fulfilled for $f=z_3$ and $g=\psi(x_2)$, since $\psi(x_2)^{\mathbf{w}}=(z_2+h_2)^{\mathbf{w}}$ does not belong to $k[z_3^{\mathbf{w}}]$, and $\psi(x_2)$ and $z_3=\psi(x_3)$ are algebraically independent over k. Since $\deg_{\mathbf{w}}\psi(x_2)=\gamma_2^{\mathbf{w}}$ and $\deg_{\mathbf{w}}z_3=\gamma_3^{\mathbf{w}}$, the condition (1) is equivalent to $\gamma_2^{\mathbf{w}}>\gamma_3^{\mathbf{w}}$.

Lemma 2.3. *Assume that* $\gamma_2^{\mathbf{w}}>\gamma_3^{\mathbf{w}}$. *Then,* ϕ *satisfies* (2.3) *if one of the following conditions holds*:
(i) $d\geq 3$ *and* $\psi(x_2)^{\mathbf{w}}$ *and* $z_3^{\mathbf{w}}$ *are algebraically independent over* k.
(ii) $d\geq 3$ *and* $\deg_{\mathbf{w}}dz_2\wedge dz_3>(d-1)\deg_{\mathbf{w}}z_3$.
(iii) $d\geq 9$ *and* $d\neq 10,12$.

Proof. First, assume that (i) is satisfied. Then, we have $k[\psi(x_2),z_3]^{\mathbf{w}}=k[\psi(x_2)^{\mathbf{w}},z_3^{\mathbf{w}}]$, since $\psi(x_2)^{\mathbf{w}}$ and $z_3^{\mathbf{w}}$ are algebraically independent over k. Take any $h\in k[\psi(x_2),z_3]$. Then, $h^{\mathbf{w}}$ belongs to $k[\psi(x_2)^{\mathbf{w}},z_3^{\mathbf{w}}]$. Hence, we know that $\theta(\psi(x_2))^{\mathbf{w}}\approx(\psi(x_2)^{\mathbf{w}})^d\not\approx h^{\mathbf{w}}z_3^{\mathbf{w}}=(hz_3)^{\mathbf{w}}$. This implies that

$$\deg_{\mathbf{w}}\bigl(\theta(\psi(x_2))+hz_3\bigr)=\max\{\deg_{\mathbf{w}}\theta(\psi(x_2)),\deg_{\mathbf{w}}hz_3\}\geq\deg_{\mathbf{w}}\theta(\psi(x_2))=d\gamma_2^{\mathbf{w}}.$$

Thus, we obtain $\gamma_1^{\mathbf{w}}\geq d\gamma_2^{\mathbf{w}}$ in view of (2.5). Since $d\geq 3$ and $\gamma_2^{\mathbf{w}}>\gamma_3^{\mathbf{w}}$ by assumption, it follows that

$$2\gamma_1^{\mathbf{w}}\geq 2d\gamma_2^{\mathbf{w}}>5\gamma_2^{\mathbf{w}}>3\gamma_2^{\mathbf{w}}+2\gamma_3^{\mathbf{w}},\quad \gamma_1^{\mathbf{w}}\geq d\gamma_2^{\mathbf{w}}\geq 3\gamma_2^{\mathbf{w}}>\gamma_2^{\mathbf{w}}+2\gamma_3^{\mathbf{w}}.$$

Therefore, ϕ satisfies (2.3).

Next, assume that (ii) or (iii) is satisfied. Since $d\psi(x_2)\wedge dz_3=dz_2\wedge dz_3$, we see that (ii) is equivalent to (ii) of Lemma 2.3. Clearly, (iii) is the same as (iii) of Lemma 2.3. Since $\gamma_2^{\mathbf{w}}>\gamma_3^{\mathbf{w}}$ by assumption, the conditions (1), (2) and (3) listed before Lemma 2.3 are fulfilled for $f=z_3$ and $g=\psi(x_2)$ as mentioned. Hence, we know by Lemma 2.3 that $\gamma_1^{\mathbf{w}}=\eta(\theta;z_3,\psi(x_2))$ is

greater than $\eta_1 = \gamma_3^{\mathbf{w}} + (3/2)\gamma_2^{\mathbf{w}}$ and $\eta_2 = 2\gamma_3^{\mathbf{w}} + \gamma_2^{\mathbf{w}}$. Therefore, we get (2.3). $\square$

Now, we prove Theorem 1.5 (i). Assume that $d \geq 9$ and $d \neq 10, 12$. Suppose to the contrary that $\deg_{\mathbf{w}} \phi(y_2) \leq \deg_{\mathbf{w}} \phi(x_3)$ for some $\phi \in G_\theta$. Since $\deg_{\mathbf{w}} \phi(x_3) = \deg_{\mathbf{w}} z_3 = \gamma_3^{\mathbf{w}}$, and $\gamma_0^{\mathbf{w}} > 0$, we have $\deg_{\mathbf{w}} \phi(y_2) \leq \gamma_3^{\mathbf{w}} < \gamma_0^{\mathbf{w}} + \gamma_3^{\mathbf{w}}$. By Lemma 2.1, it follows that $\gamma_3^{\mathbf{w}} < \gamma_2^{\mathbf{w}}$ and $\gamma_0^{\mathbf{w}} < \gamma_2^{\mathbf{w}}$. Hence, ϕ satisfies the assumption of Lemma 2.3. Since $d \geq 9$ and $d \neq 10, 12$, we know by (iii) that ϕ satisfies (2.3). Since $\gamma_0^{\mathbf{w}} < \gamma_2^{\mathbf{w}}$, (2.3) implies that $\gamma_0^{\mathbf{w}} < \gamma_1^{\mathbf{w}}$. Thus, we conclude from Lemma 2.2 that ϕ is wild, a contradiction. Therefore, we have $\deg_{\mathbf{w}} \phi(y_2) > \deg_{\mathbf{w}} \phi(x_3)$ for each $\phi \in G_\theta$. This completes the proof of Theorem 1.5 (i).

3. Proof (II)

The goal of this section is to prove the following proposition, which is a key to the proof of Theorem 1.5 (ii).

Proposition 3.1. *Assume that* $d \geq 9$. *Let* $\phi \in G_\theta$ *be such that* $\deg_{\mathbf{v}} \phi(y_2) > \deg_{\mathbf{v}} \phi(x_3)$. *Then, the following assertions hold*:
(i) $\deg_{\mathbf{v}} \phi(y_2) = 3\mathbf{e}_1$.
(ii) $\phi(x_3) = z_3 = \alpha_3 x_3 + g_3$ *for some* $\alpha_3 \in k^\times$ *and* $g_3 \in k[x_2]$.
(iii) $\gamma_2^{\mathbf{v}} < \mathbf{e}_1$.

We begin with the following lemma, which is proved by a technique similar to Hadas–Makar-Limanov [**11**, Corollary 3.3].

Lemma 3.2. *Let* τ *be an element of* $\mathrm{Aut}(k[\mathbf{x}]/k)$, *and* $\mathbf{w}$ *an element of* Γ^n, *where* $n \geq 3$ *may be arbitrary. Assume that there exist* $i_1, i_2 \in \{1, \ldots, n\}$ *such that* $\tau(x_{i_1})^{\mathbf{w}}$ *is divisible by* x_i *for each* $i \neq i_2$. *Then,* $\tau(x_j)^{\mathbf{w}}$ *belongs to* $k[\mathbf{x} \setminus \{x_{i_2}\}]$ *for* $j = 1, \ldots, n$.

Proof. For each $0 \neq D \in \mathrm{Der}_k\, k[\mathbf{x}]$, we define

$$\gamma_D = \max\{\deg_{\mathbf{w}} D(x_i) x_i^{-1} \mid i = 1, \ldots, n\},$$

and $D^{\mathbf{w}} \in \mathrm{Der}_k\, k[\mathbf{x}]$ by

$$D^{\mathbf{w}}(x_i) = \begin{cases} D(x_i)^{\mathbf{w}} & \text{if} \quad \deg_{\mathbf{w}} D(x_i) x_i^{-1} = \gamma_D \\ 0 & \text{if} \quad \deg_{\mathbf{w}} D(x_i) x_i^{-1} < \gamma_D \end{cases} \tag{3.1}$$

for $i = 1, \ldots, n$. We show that $D^{\mathbf{w}}(f^{\mathbf{w}}) \neq 0$ implies $D(f)^{\mathbf{w}} = D^{\mathbf{w}}(f^{\mathbf{w}})$ for each $f \in k[\mathbf{x}] \setminus \{0\}$. For each $h \in k[\mathbf{x}]$ and $i = 1, \ldots, n$, we denote $h_{x_i} = \partial h / \partial x_i$ for simplicity. Then, we have $\deg_{\mathbf{w}} f_{x_i} \leq \deg_{\mathbf{w}} f x_i^{-1}$, and $\deg_{\mathbf{w}} f_{x_i} = \deg_{\mathbf{w}} f x_i^{-1}$ if and only if $f^{\mathbf{w}}$ does not belong to $k[\mathbf{x} \setminus \{x_i\}]$. Hence, we know that

$$(f^{\mathbf{w}})_{x_i} = \begin{cases} (f_{x_i})^{\mathbf{w}} & \text{if} \quad \deg_{\mathbf{w}} f_{x_i} = \deg_{\mathbf{w}} f x_i^{-1} \\ 0 & \text{if} \quad \deg_{\mathbf{w}} f_{x_i} < \deg_{\mathbf{w}} f x_i^{-1} \end{cases} \tag{3.2}$$

for $i = 1, \ldots, n$. Let I be the set of $i \in \{1, \ldots, n\}$ such that $\deg_{\mathbf{w}} D(x_i) x_i^{-1} = \gamma_D$ and $\deg_{\mathbf{w}} f_{x_i} = \deg_{\mathbf{w}} f x_i^{-1}$. Then, we have
(3.3)

$$D^{\mathbf{w}}(f^{\mathbf{w}}) = \sum_{i=1}^{n} D^{\mathbf{w}}(x_i)(f^{\mathbf{w}})_{x_i} = \sum_{i \in I} D(x_i)^{\mathbf{w}} (f_{x_i})^{\mathbf{w}} = \sum_{i \in I} \bigl(D(x_i) f_{x_i}\bigr)^{\mathbf{w}}.$$

Note that

$$\deg_{\mathbf{w}} D(x_i)f_{x_i} \le (\gamma_D + \deg_{\mathbf{w}} x_i) + (\deg_{\mathbf{w}} f - \deg_{\mathbf{w}} x_i) = \gamma_D + \deg_{\mathbf{w}} f$$

for $i = 1, \ldots, n$, in which the equality holds if and only if i belongs to I. Since $D^{\mathbf{w}}(f^{\mathbf{w}}) \neq 0$ by assumption, this implies that

$$\sum_{i \in I} \bigl(D(x_i)f_{x_i}\bigr)^{\mathbf{w}} = \left(\sum_{i=1}^{n} D(x_i)f_{x_i}\right)^{\mathbf{w}} = D(f)^{\mathbf{w}}.$$

Thus, we get $D(f)^{\mathbf{w}} = D^{\mathbf{w}}(f^{\mathbf{w}})$. Using this, we can prove by induction on l that $(D^{\mathbf{w}})^l(f^{\mathbf{w}}) \neq 0$ implies $D^l(f)^{\mathbf{w}} = (D^{\mathbf{w}})^l(f^{\mathbf{w}})$ for each $l \in \mathbf{N}$ and $f \in k[\mathbf{x}] \setminus \{0\}$. Therefore, if D is locally nilpotent, then $D^{\mathbf{w}}$ is also locally nilpotent.

Now, take any $i_0 \in \{1, \ldots, n\} \setminus \{i_1\}$, and put $D = \tau \circ (\partial/\partial x_{i_0}) \circ \tau^{-1}$. Then, D is locally nilpotent. Hence, $D^{\mathbf{w}}$ is also locally nilpotent. Thus, $\ker D^{\mathbf{w}}$ is factorially closed in $k[\mathbf{x}]$, i.e., if fg belongs to $\ker D^{\mathbf{w}}$ for $f, g \in k[\mathbf{x}] \setminus \{0\}$, then f and g belong to $\ker D^{\mathbf{w}}$ (cf. [**20**, Lemma 1.3.1]). Since $i_1 \neq i_0$, we have $D\bigl(\tau(x_{i_1})\bigr) = 0$. This implies that $D^{\mathbf{w}}(\tau(x_{i_1})^{\mathbf{w}}) = 0$, for otherwise $D(\tau(x_{i_1}))^{\mathbf{w}} = D^{\mathbf{w}}(\tau(x_{i_1})^{\mathbf{w}}) \neq 0$, a contradiction. Hence, $\tau(x_{i_1})^{\mathbf{w}}$ belongs to $\ker D^{\mathbf{w}}$. Since x_i is a factor of $\tau(x_{i_1})^{\mathbf{w}}$ for each $i \neq i_2$ by assumption, it follows that x_i belongs to $\ker D^{\mathbf{w}}$ for each $i \neq i_2$ by the factorially closedness of $\ker D^{\mathbf{w}}$. Hence, $k[\mathbf{x} \setminus \{x_{i_2}\}]$ is contained in $\ker D^{\mathbf{w}}$. Since D is nonzero, so is $D^{\mathbf{w}}$. Hence, the transcendence degree of $\ker D^{\mathbf{w}}$ over k is less than n. Thus, we conclude that $\ker D^{\mathbf{w}} = k[\mathbf{x} \setminus \{x_{i_2}\}]$. If $j \neq i_0$, then we have $D\bigl(\tau(x_j)\bigr) = 0$. This implies that $D^{\mathbf{w}}\bigl(\tau(x_j)^{\mathbf{w}}\bigr) = 0$ as mentioned. Hence, $\tau(x_j)^{\mathbf{w}}$ belongs to $\ker D^{\mathbf{w}} = k[\mathbf{x} \setminus \{x_{i_2}\}]$. Since $n \ge 3$ by assumption, we may take $i_0' \in \{1, \ldots, n\} \setminus \{i_0, i_1\}$. Then, by a similar argument with i_0 replaced by i_0', we can verify that $\tau(x_j)^{\mathbf{w}}$ belongs to $k[\mathbf{x} \setminus \{x_{i_2}\}]$ for each $j \neq i_0'$. Thus, $\tau(x_{i_0})^{\mathbf{w}}$ also belongs to $k[\mathbf{x} \setminus \{x_{i_2}\}]$. Therefore, $\tau(x_j)^{\mathbf{w}}$ belongs to $k[\mathbf{x} \setminus \{x_{i_2}\}]$ for $j = 1, \ldots, n$. $\square$

Recall that $f_\theta = x_1x_3 + \theta(x_2)$, $y_2 = x_2 + f_\theta x_3$ and $y_3 = x_3$. Since $\mathbf{v} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1)$, we have

$$f_\theta^{\mathbf{v}} = x_1x_3, \quad y_2^{\mathbf{v}} = x_1x_3^2, \quad y_3^{\mathbf{v}} = x_3 \tag{3.4}$$

by the definition of Λ. Hence, we see from (1.1) that

$$y_1^{\mathbf{v}} = -\theta(y_2)^{\mathbf{v}}x_3^{-1} \approx x_1^d x_3^{2d-1}. \tag{3.5}$$

To prove Proposition 3.1, assume that $d \ge 9$, and take $\phi \in G_\theta$ such that $\deg_{\mathbf{v}} \phi(y_2)$ is greater than $\deg_{\mathbf{v}} \phi(x_3) = \deg_{\mathbf{v}} z_3$. Then, we have $\phi(y_1) = y_1$. Hence, $\phi(y_1)^{\mathbf{v}} = y_1^{\mathbf{v}}$ is divisible by x_1 and x_3 by (3.5). Since $\phi(y_1) = (\phi \circ \sigma_\theta)(x_1)$, it follows that $\phi(y_2)^{\mathbf{v}} = (\phi \circ \sigma_\theta)(x_2)^{\mathbf{v}}$ and $z_3^{\mathbf{v}} = (\phi \circ \sigma_\theta)(x_3)^{\mathbf{v}}$ belong to $k[x_1, x_3]$ by Lemma 3.2. Hence, we know that $\deg_{\mathbf{v}} \phi(y_2) = a\mathbf{e}_1$ and $\deg_{\mathbf{v}} z_3 = b\mathbf{e}_1$ for some $a, b \in \mathbf{N}$. Since $\deg_{\mathbf{v}} \phi(y_2) > \deg_{\mathbf{v}} z_3$ by assumption, we get $a \ge b + 1$.

Lemma 3.3. *If* $\deg_{\mathbf{v}} y_1z_3 = d \deg_{\mathbf{v}} \phi(y_2)$, *then we have*

$$(a, b) = (3, 1), \quad \deg_{\mathbf{v}} \phi(y_2) = 3 \deg_{\mathbf{v}} z_3$$

and $z_3^{\mathbf{v}}$ *is a linear form in* x_1 *and* x_3 *over* k.

PROOF. By (3.5), we see that $\deg_{\mathbf{v}} y_1 = (3d-1)\mathbf{e}_1$. Since $\deg_{\mathbf{v}} \phi(y_2) = a\mathbf{e}_1$, $\deg_{\mathbf{v}} z_3 = b\mathbf{e}_1$, and $\deg_{\mathbf{v}} y_1 z_3 = d \deg_{\mathbf{v}} \phi(y_2)$ by assumption, we know that $3d-1+b = da$. Because $1 \le b \le a-1$, it follows that $1 \le da-3d+1 \le a-1$. Hence, we have

$$3 \le a \le \frac{3d-2}{d-1} = 3 + \frac{1}{d-1} < 4,$$

since $d \ge 9$. Thus, we get $a = 3$. Since $3d-1+b = da = 3d$, it follows that $b = 1$. Therefore, we have $\deg_{\mathbf{v}} \phi(y_2) = 3\mathbf{e}_1 = 3 \deg_{\mathbf{v}} z_3$. Since $\deg_{\mathbf{v}} z_3 = \mathbf{e}_1$ and $\mathbf{v} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1)$, we know that $z_3^{\mathbf{v}}$ is a linear form in x_1 and x_3 over k. □

Using this lemma, we can prove the following proposition.

Proposition 3.4. *If $(y_1 z_3)^{\mathbf{v}} \approx (\phi(y_2)^{\mathbf{v}})^d$, then we have $\phi(y_2)^{\mathbf{v}} \approx x_1 x_3^2$ and $z_3^{\mathbf{v}} \approx x_3$.*

PROOF. Since $(y_1 z_3)^{\mathbf{v}} \approx (\phi(y_2)^{\mathbf{v}})^d$, we have $\deg_{\mathbf{v}} y_1 z_3 = d \deg_{\mathbf{v}} \phi(y_2)$. Hence, we may write $z_3^{\mathbf{v}} = \alpha x_1 + \beta x_3$ by Lemma 3.3, where $\alpha, \beta \in k$ are such that $\alpha \neq 0$ or $\beta \neq 0$. In view of (3.5), we have

$$(\phi(y_2)^{\mathbf{v}})^d \approx (y_1 z_3)^{\mathbf{v}} \approx x_1^d x_3^{2d-1}(\alpha x_1 + \beta x_3).$$

Since $d \ge 9$, this implies that $\alpha = 0$, $\beta \neq 0$, $\phi(y_2)^{\mathbf{v}} \approx x_1 x_3^2$ and $z_3^{\mathbf{v}} = \beta x_3 \approx x_3$. □

Since $\sigma_\theta(f_\theta) = f_\theta$, $\sigma_\theta(x_3) = x_3$ and $\phi(y_1) = y_1$, we have

$$\phi(f_\theta) = \phi(\sigma_\theta(f_\theta)) = \phi(y_1 x_3 + \theta(y_2)) = y_1 z_3 + \theta(\phi(y_2)). \tag{3.6}$$

The following proposition forms the core of the proof of Proposition 3.1.

Proposition 3.5. *If $(y_1 z_3)^{\mathbf{v}} \not\approx (\phi(y_2)^{\mathbf{v}})^d$, then the following inequalities hold*:

(i) $\deg_{\mathbf{v}} d\phi(f_\theta) \wedge dz_3 > (d-1) \deg_{\mathbf{v}} \phi(y_2)$.
(ii) $\deg_{\mathbf{v}} dz_2 \wedge dz_3 > (d-1) \deg_{\mathbf{v}} \phi(y_2) + \deg_{\mathbf{v}} z_3$.
(iii) $\gamma_0^{\mathbf{v}} + \gamma_3^{\mathbf{v}} > (d-1) \deg_{\mathbf{v}} \phi(y_2)$.

PROOF. (i) First, assume that $\deg_{\mathbf{v}} y_1 z_3 = d \deg_{\mathbf{v}} \phi(y_2)$. Then, we have

$$\phi(f_\theta)^{\mathbf{v}} = \bigl(y_1 z_3 + \theta(\phi(y_2))\bigr)^{\mathbf{v}} = (y_1 z_3)^{\mathbf{v}} + c(\phi(y_2)^{\mathbf{v}})^d \tag{3.7}$$

by (3.6), since $(y_1 z_3)^{\mathbf{v}} \not\approx (\phi(y_2)^{\mathbf{v}})^d$ by assumption. Hence, we get

$$\deg_{\mathbf{v}} \phi(f_\theta) = d \deg_{\mathbf{v}} \phi(y_2). \tag{3.8}$$

Now, suppose that (i) is false. Then, we have

$$\deg_{\mathbf{v}} d\phi(f_\theta) \wedge dz_3 \le (d-1) \deg_{\mathbf{v}} \phi(y_2) < \deg_{\mathbf{v}} \phi(f_\theta) + \deg_{\mathbf{v}} z_3$$

by (3.8). This implies that $\phi(f_\theta)^{\mathbf{v}}$ and $z_3^{\mathbf{v}}$ are algebraically dependent over k. Since $\phi(f_\theta)^{\mathbf{v}}$ and $z_3^{\mathbf{v}}$ are $\mathbf{v}$-homogeneous polynomials of positive $\mathbf{v}$-degrees, it follows that $(\phi(f_\theta)^{\mathbf{v}})^l = c_1 (z_3^{\mathbf{v}})^m$ for some $l, m \in \mathbf{N}$ with $\gcd(l, m) = 1$ and $c_1 \in k^\times$. By (3.8) and Lemma 3.3, we know that $\deg_{\mathbf{v}} \phi(f_\theta) = 3d \deg_{\mathbf{v}} z_3$. Hence, we get $(l, m) = (1, 3d)$. By (3.7), it follows that $c(\phi(y_2)^{\mathbf{v}})^d = c_1 (z_3^{\mathbf{v}})^{3d} - y_1^{\mathbf{v}} z_3^{\mathbf{v}}$. From this, we see that $(\phi(y_2)^{\mathbf{v}})^d$ is divisible by $z_3^{\mathbf{v}}$. By Lemma 3.3, $z_3^{\mathbf{v}}$ is a linear form, and so irreducible. Hence, $z_3^{\mathbf{v}}$ divides $\phi(y_2)^{\mathbf{v}}$. Since $d \ge 9$ by assumption, it follows that $z_3^{\mathbf{v}}$

divides $y_1^{\mathbf{v}}$. By (3.5), we see that $z_3^{\mathbf{v}} = c_2 x_i$ for some $c_2 \in k^\times$ and $i \in \{1, 3\}$. Thus, $c(\phi(y_2)^{\mathbf{v}})^d = c_1(c_2 x_i)^{3d} - c_2 x_i y_1^{\mathbf{v}}$ is a binomial. Since k is of characteristic zero, no binomial is a proper power of a polynomial. Therefore, we get $d = 1$, a contradiction.

Next, assume that $\deg_{\mathbf{v}} y_1 z_3 < d \deg_{\mathbf{v}} \phi(y_2)$. Then, we have $3d - 1 + b < da$. This yields that $a > 3 + (b-1)/d \geq 3$. Hence, we get $a \geq 4$. First, we prove that

$$\deg_{\mathbf{v}} y_1 z_3^2 \leq (d-1) \deg_{\mathbf{v}} \phi(y_2) \tag{3.9}$$

by contradiction. Suppose that (3.9) is false. Then, we have

$$(d-1)a \leq (3d - 1 + 2b) - 1.$$

Since $b \leq a - 1$, it follows that $(d-1)a \leq 3d + 2a - 4$. Since $d \geq 9$ by assumption, this yields that

$$a \leq \frac{3d-4}{d-3} = 3 + \frac{5}{d-3} < 4,$$

a contradiction. Therefore, (3.9) is true. By (3.6), we get

$$d\phi(f_\theta) \wedge dz_3 = d(y_1 z_3) \wedge dz_3 + d\theta(\phi(y_2)) \wedge dz_3. \tag{3.10}$$

Since $\phi(y_2) = \phi(\sigma_\theta(x_2))$ and $z_3 = \phi(\sigma_\theta(x_3))$ are algebraically independent over k, we know that $d\phi(y_2) \wedge dz_3$ is nonzero. Hence, the $\mathbf{v}$-degree of $d\theta(\phi(y_2)) \wedge dz_3 = \theta'(\phi(y_2)) d\phi(y_2) \wedge dz_3$ is greater than $\deg_{\mathbf{v}} \theta'(\phi(y_2)) = (d-1) \deg_{\mathbf{v}} \phi(y_2)$. Thus, it follows from (3.9) that

$$\deg_{\mathbf{v}} d\theta(\phi(y_2)) \wedge dz_3 > (d-1) \deg_{\mathbf{v}} \phi(y_2) \geq \deg_{\mathbf{v}} y_1 z_3^2 \geq \deg_{\mathbf{v}} d(y_1 z_3) \wedge dz_3.$$

In view of (3.10), this implies that $\deg_{\mathbf{v}} d\phi(f_\theta) \wedge dz_3 > (d-1) \deg_{\mathbf{v}} \phi(y_2)$.

Finally, assume that $\deg_{\mathbf{v}} y_1 z_3 > d \deg_{\mathbf{v}} \phi(y_2)$. Then, we have

$$\begin{aligned}\deg_{\mathbf{v}} d\theta(\phi(y_2)) \wedge dz_3 &\leq \deg_{\mathbf{v}} \theta(\phi(y_2)) + \deg_{\mathbf{v}} z_3 = d \deg_{\mathbf{v}} \phi(y_2) + \deg_{\mathbf{v}} z_3 \\ &< \deg_{\mathbf{v}} y_1 z_3 + \deg_{\mathbf{v}} z_3 = \deg_{\mathbf{v}} y_1 z_3^2.\end{aligned}$$

We show that

$$\deg_{\mathbf{v}} y_1 z_3^2 = \deg_{\mathbf{v}} d(y_1 z_3) \wedge dz_3. \tag{3.11}$$

Then, it follows that $\deg_{\mathbf{v}} d\theta(\phi(y_2)) \wedge dz_3 < \deg_{\mathbf{v}} d(y_1 z_3) \wedge dz_3$. In view of (3.10), this implies that $\deg_{\mathbf{v}} d\phi(f_\theta) \wedge dz_3 = \deg_{\mathbf{v}} d(y_1 z_3) \wedge dz_3$. By (3.11), this is equal to $\deg_{\mathbf{v}} y_1 z_3^2$, and is greater than $(d-1) \deg_{\mathbf{v}} \phi(y_2)$ by the assumption that $\deg_{\mathbf{v}} y_1 z_3 > d \deg_{\mathbf{v}} \phi(y_2)$. Therefore, we get $\deg_{\mathbf{v}} d\phi(f_\theta) \wedge dz_3 > (d-1) \deg_{\mathbf{v}} \phi(y_2)$.

Since $d(y_1 z_3) \wedge dz_3 = z_3 dy_1 \wedge dz_3$, it suffices to verify that $y_1^{\mathbf{v}}$ and $z_3^{\mathbf{v}}$ are algebraically independent over k. Suppose to the contrary that $y_1^{\mathbf{v}}$ and $z_3^{\mathbf{v}}$ are algebraically dependent over k. Then, we have $(y_1^{\mathbf{v}})^q \approx (z_3^{\mathbf{v}})^r$ for some $q, r \in \mathbf{N}$ with $\gcd(q, r) = 1$, since $y_1^{\mathbf{v}}$ and $z_3^{\mathbf{v}}$ are $\mathbf{v}$-homogeneous polynomials of positive $\mathbf{v}$-degrees. Because $\gcd(d, 2d-1) = 1$, we see that $y_1^{\mathbf{v}} \approx x_1^d x_2^{2d-1}$ is not a proper power of a polynomial. Hence, we know that $r = 1$. Thus, we get $q \deg_{\mathbf{v}} y_1 = \deg_{\mathbf{v}} z_3$. Since $\deg_{\mathbf{v}} y_1 z_3 > d \deg_{\mathbf{v}} \phi(y_2)$ by assumption, and $\deg_{\mathbf{v}} \phi(y_2) > \deg_{\mathbf{v}} z_3$ by the choice of ϕ, it follows that

$$2 \deg_{\mathbf{v}} z_3 = q \deg_{\mathbf{v}} y_1 + \deg_{\mathbf{v}} z_3 \geq \deg_{\mathbf{v}} y_1 z_3 > d \deg_{\mathbf{v}} \phi(y_2) > d \deg_{\mathbf{v}} z_3.$$

This contradicts that $d \geq 9$. Therefore, $y_1^{\mathbf{v}}$ and $z_3^{\mathbf{v}}$ are algebraically independent over k, proving that $\deg_{\mathbf{v}} d\phi(f_\theta) \wedge dz_3 > (d-1)\deg_{\mathbf{v}} \phi(y_2)$.

(ii) By (i), we know that

$$\begin{aligned}\deg_{\mathbf{v}} z_3 d\phi(f_\theta) \wedge dz_3 &= \deg_{\mathbf{v}} d\phi(f_\theta) \wedge dz_3 + \deg_{\mathbf{v}} z_3 \\ &> (d-1)\deg_{\mathbf{v}} \phi(y_2) + \deg_{\mathbf{v}} z_3 > \deg_{\mathbf{v}} d\phi(y_2) \wedge dz_3,\end{aligned}$$

since $d \geq 9$. Because $z_2 = \phi(y_2) - \phi(f_\theta)z_3$ by (2.1), we have

$$dz_2 \wedge dz_3 = d\phi(y_2) \wedge dz_3 - z_3 d\phi(f_\theta) \wedge dz_3.$$

Therefore, we get

$$\deg_{\mathbf{v}} dz_2 \wedge dz_3 = \deg_{\mathbf{v}} z_3 d\phi(f_\theta) \wedge dz_3 > (d-1)\deg_{\mathbf{v}} \phi(y_2) + \deg_{\mathbf{v}} z_3.$$

(iii) Take $h_0 \in k[z_3]$ such that $\gamma_0^{\mathbf{v}} = \deg_{\mathbf{v}}(\phi(f_\theta) + h_0)$. Then, we have

$$\gamma_0^{\mathbf{v}} + \gamma_3^{\mathbf{v}} = \deg_{\mathbf{v}}(\phi(f_\theta) + h_0) + \deg_{\mathbf{v}} z_3 \geq \deg_{\mathbf{v}} d(\phi(f_\theta) + h_0) \wedge dz_3 = \deg_{\mathbf{v}} d\phi(f_\theta) \wedge dz_3.$$

Therefore, we get $\gamma_0^{\mathbf{v}} + \gamma_3^{\mathbf{v}} > (d-1)\deg_{\mathbf{v}} \phi(y_2)$ by (i). $\square$

Now, let us complete the proof of Proposition 3.1. First, we prove that $(y_1 z_3)^{\mathbf{v}} \approx (\phi(y_2)^{\mathbf{v}})^d$. Suppose to the contrary that $(y_1 z_3)^{\mathbf{v}} \not\approx (\phi(y_2)^{\mathbf{v}})^d$. Then, we have the three inequalities (i), (ii) and (iii) of Proposition 3.5. We deduce that ϕ is wild by means of Lemma 2.2. By the inequality (iii), we have

$$\gamma_0^{\mathbf{v}} + \gamma_3^{\mathbf{v}} > (d-1)\deg_{\mathbf{v}} \phi(y_2) > \deg_{\mathbf{v}} \phi(y_2),$$

since $d \geq 9$. By Lemma 2.1, it follows that $\gamma_3^{\mathbf{w}} < \gamma_2^{\mathbf{w}}$ and $\gamma_0^{\mathbf{w}} < \gamma_2^{\mathbf{w}}$. Hence, ϕ satisfies the assumption of Lemma 2.3. Since $\deg_{\mathbf{v}} \phi(y_2) > \deg_{\mathbf{v}} z_3$ by the choice of ϕ, the inequality (ii) yields that

$$\deg_{\mathbf{v}} dz_2 \wedge dz_3 > (d-1)\deg_{\mathbf{v}} \phi(y_2) + \deg_{\mathbf{v}} z_3 > (d-1)\deg_{\mathbf{v}} z_3.$$

Since $d \geq 9$, this implies that ϕ satisfies (2.3) because of Lemma 2.3 (ii). The second part of (2.3) implies $\gamma_1^{\mathbf{v}} > \gamma_2^{\mathbf{v}}$. Since $\gamma_0^{\mathbf{v}} < \gamma_2^{\mathbf{v}}$, it follows that $\gamma_0^{\mathbf{v}} < \gamma_1^{\mathbf{v}}$. Thus, we conclude from Lemma 2.2 that ϕ is wild, a contradiction. This proves that $(y_1 z_3)^{\mathbf{v}} \approx (\phi(y_2)^{\mathbf{v}})^d$. Therefore, we get $\phi(y_2)^{\mathbf{v}} \approx x_1 x_3^2$ and $z_3^{\mathbf{v}} \approx x_3$ thanks to Proposition 3.4. Hence, we have $\deg_{\mathbf{v}} \phi(y_2) = 3\mathbf{e}_1$, proving Proposition 3.1 (i). Since $\mathbf{v} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1)$, we know that $z_3 = \alpha_3 x_3 + g_3$ for some $\alpha_3 \in k^\times$ and $g_3 \in k[x_2]$. This proves Proposition 3.1 (ii).

Next, we prove Proposition 3.1 (iii). First, we show that $\gamma_2^{\mathbf{v}} \leq \mathbf{e}_1$. Suppose to the contrary that $\gamma_2^{\mathbf{v}} > \mathbf{e}_1$. We deduce that ϕ is wild by means of Lemma 2.2. Since $\gamma_3^{\mathbf{v}} = \deg_{\mathbf{v}} z_3 = \mathbf{e}_1$, we have $\gamma_2^{\mathbf{v}} > \gamma_3^{\mathbf{v}}$. Hence, ϕ satisfies the assumption of Lemma 2.3. Define $\psi \in \mathrm{Aut}(k[\mathbf{x}]/k)$ as before Lemma 2.2. Then, $\psi(x_2)^{\mathbf{v}} = (z_2 + h_2)^{\mathbf{v}}$ does not belong to $k[z_3^{\mathbf{v}}] = k[x_3]$ by the minimality of $\gamma_2^{\mathbf{v}}$. Hence, $\psi(x_2)^{\mathbf{v}}$ and $z_3^{\mathbf{v}} \approx x_3$ are algebraically independent over k. Since $d \geq 9$, we know by Lemma 2.3 (i) that ϕ satisfies (2.3). Finally, we show that $\gamma_0^{\mathbf{v}} < \gamma_1^{\mathbf{v}}$. The second part of (2.3) implies $\gamma_1^{\mathbf{v}} > \gamma_2^{\mathbf{v}}$. If $\gamma_0^{\mathbf{v}} \leq \gamma_2^{\mathbf{v}}$, then we get $\gamma_0^{\mathbf{v}} < \gamma_1^{\mathbf{v}}$. So assume that $\gamma_0^{\mathbf{v}} > \gamma_2^{\mathbf{v}}$. Then, we have $\deg_{\mathbf{v}} \phi(y_2) \geq \gamma_0^{\mathbf{v}} + \gamma_3^{\mathbf{v}}$ by the contraposition of Lemma 2.1. Since $\deg_{\mathbf{v}} \phi(y_2) = 3\mathbf{e}_1$ and $\gamma_3^{\mathbf{v}} = \mathbf{e}_1$, it follows that $2\mathbf{e}_1 \geq \gamma_0^{\mathbf{v}}$. Hence, we get

$$\gamma_1^{\mathbf{v}} \geq \gamma_2^{\mathbf{v}} + 2\gamma_3^{\mathbf{v}} > 2\gamma_3^{\mathbf{v}} = 2\mathbf{e}_1 \geq \gamma_0^{\mathbf{v}}$$

by the second part of (2.3). Thus, we conclude from Lemma 2.2 that ϕ is wild, a contradiction. This proves that $\gamma_2^{\mathbf{v}} \leq \mathbf{e}_1$. Therefore, we may write

$$z_2 + h_2 = \alpha x_1 + \alpha' x_3 + g,$$

where $\alpha, \alpha' \in k$ and $g \in k[x_2]$. In order to conclude that $\gamma_2^{\mathbf{v}} < \mathbf{e}_1$, it suffices to show that $(\alpha, \alpha') = (0, 0)$. Suppose to the contrary that $(\alpha, \alpha') \neq (0, 0)$. Then, we have $\alpha \neq 0$, since $(z_2 + h_2)^{\mathbf{v}} = \alpha x_1 + \alpha' x_3$ does not belong to $k[z_3^{\mathbf{v}}] = k[x_3]$. Since h_2 is an element of $k[z_3]$, and $z_3 = \alpha_3 x_3 + g_3$ belongs to $k[x_2, x_3]$, we know that h_2 belongs to $k[x_2, x_3]$. Hence, $z_2 = \alpha x_1 + \alpha' x_3 + g - h_2$ is a linear polynomial in x_1 over $k[x_2, x_3]$ with leading coefficient α. Regard $z_1 z_3$ and $\theta(z_2)$ as polynomials in x_1 over $k[x_2, x_3]$. Then, the leading coefficient of $z_1 z_3$ is a multiple of z_3, while that of $\theta(z_2)$ is an element of $k^\times$. Since z_3 does not belong to k, it follows that

$$\deg_{x_1} \phi(f_\theta) = \deg_{x_1} \bigl(z_1 z_3 + \theta(z_2)\bigr) = \max\{\deg_{x_1} z_1 z_3, \deg_{x_1} \theta(z_2)\}.$$

Since $\deg_{x_1} z_3 = 0$ and $\deg_{x_1} z_2 = 1$, this gives that $\deg_{x_1} \phi(f_\theta) = \max\{\deg_{x_1} z_1, d\}$. Hence, we get

$$\deg_{x_1} \phi(y_2) = \deg_{x_1} \bigl(z_2 + \phi(f_\theta) z_3\bigr) = \deg_{x_1} \phi(f_\theta) = \max\{\deg_{x_1} z_1, d\}.$$

Consequently, we see from (1.1) that

$$\begin{aligned} \deg_{x_1} \phi(y_1) &= \deg_{x_1} \Bigl(z_1 + \bigl(\theta(z_2) - \theta(\phi(y_2))\bigr) z_3^{-1}\Bigr) \\ &= \deg_{x_1} \theta(\phi(y_2)) = d \deg_{x_1} \phi(y_2) \geq d^2. \end{aligned}$$

On the other hand, we have $\deg_{x_1} y_2 = \deg_{x_1} (x_2 + f_\theta x_3) = 1$. Hence, we get

$$\deg_{x_1} y_1 = \deg_{x_1} \Bigl(x_1 + \bigl(\theta(x_2) - \theta(y_2)\bigr) x_3^{-1}\Bigr) = \deg_{x_1} \theta(y_2) = d.$$

This contradicts that $\phi(y_1) = y_1$, thus proving that $(\alpha, \alpha') = (0, 0)$. Therefore, we conclude that $\gamma_2^{\mathbf{v}} < \mathbf{e}_1$. This completes the proof of Proposition 3.1 (iii).

4. Proof (III)

In this section, we complete the proof of Theorem 1.5 (ii). Assume that $d \geq 9$. Take $\phi \in G_\theta$ such that $\deg_{\mathbf{v}} \phi(y_2) > \deg_{\mathbf{v}} \phi(x_3)$. Then, we have

$$\phi(x_3) = z_3 = \alpha_3 x_3 + g_3 \tag{4.1}$$

for some $\alpha_3 \in k^\times$ and $g_3 \in k[x_2]$ by Proposition 3.1 (ii). We establish that $\phi = \phi_{\alpha_3}$ and α_3 belongs to T_θ. If $\phi = \mathrm{id}_{k[\mathbf{x}]}$, then we have $\alpha_3 = 1$. Since $\iota : T_\theta \to G_\theta$ is a homomorphism of groups, it follows that $\phi_{\alpha_3} = \rho(1) = \mathrm{id}_{k[\mathbf{x}]}$. Hence, the assertion is true. In what follows, we assume that $\phi \neq \mathrm{id}_{k[\mathbf{x}]}$.

By Proposition 3.1 (iii), $\deg_{\mathbf{v}}(z_2 + h_2) = \gamma_2^{\mathbf{v}}$ is less than $\mathbf{e}_1$. Hence, $z_2 + h_2$ belongs to $k[x_2]$. Since $z_2 + h_2 = \psi(x_2)$ is a coordinate of $k[\mathbf{x}]$ over k, it follows that $z_2 + h_2$ is a linear polynomial in x_2 over k. Hence, we have

$$\phi(x_2) = z_2 = \alpha_2 x_2 + g_2 \tag{4.2}$$

for some $\alpha_2 \in k^\times$ and $g_2 \in k[z_3] = k[\alpha_3 x_3 + g_3]$, since h_2 is an element of $k[z_3]$. From this and (4.1), we see that ϕ induces an automorphism of $k[x_2, x_3]$. Thus, we know that $k[x_1, \phi(x_2), \phi(x_3)] = k[\mathbf{x}]$. This implies that

$$\phi(x_1) = \alpha_1 x_1 + g_1 \tag{4.3}$$

for some $\alpha_1 \in k^\times$ and $g_1 \in k[\phi(x_2), \phi(x_3)] = k[x_2, x_3]$ by the following lemma.

Lemma 4.1. *Let $\tau \in \mathrm{Aut}(k[\mathbf{x}]/k)$ be such that $k[x_i, \tau(x_2), \tau(x_3)] = k[\mathbf{x}]$ for some $i \in \{1, 2, 3\}$. Then, we have $\tau(x_1) = \alpha x_i + g$ for some $\alpha \in k^\times$ and $g \in k[\tau(x_2), \tau(x_3)]$.*

Proof. Since $k[x_i, \tau(x_2), \tau(x_3)] = k[\mathbf{x}]$, we can define $\rho \in \mathrm{Aut}(k[\mathbf{x}]/k)$ by

$$\rho(x_1) = x_i \quad \text{and} \quad \rho(x_j) = \tau(x_j) \quad \text{for} \quad j = 2, 3.$$

Then, $\rho^{-1} \circ \tau$ belongs to $\mathrm{Aut}(k[\mathbf{x}]/k[x_2, x_3])$. Hence, we may write $(\rho^{-1} \circ \tau)(x_1) = \alpha x_1 + g'$, where $\alpha \in k^\times$ and $g' \in k[x_2, x_3]$. Then, we have

$$\tau(x_1) = \rho((\rho^{-1} \circ \tau)(x_1)) = \rho(\alpha x_1 + g') = \alpha x_i + \rho(g'),$$

in which $\rho(g')$ is an element of $\rho(k[x_2, x_3]) = k[\tau(x_2), \tau(x_3)]$. $\square$

Since $y_2 = x_2 + f_\theta x_3 = x_2 + \big(x_1 x_3 + \theta(x_2)\big)x_3$ is a linear polynomial in x_1 with leading coefficient x_3^2, we see from (1.1) that

$$y_1 = -c x_1^d x_3^{2d-1} + (\text{terms of lower degree in } x_1).$$

In view of (4.1), (4.2) and (4.3), we have

$$\phi(y_1) = -c(\alpha_1 x_1)^d (\alpha_3 x_3 + g_3)^{2d-1} + (\text{terms of lower degree in } x_1).$$

Since $\phi(y_1) = y_1$, we get $\alpha_1^d (\alpha_3 x_3 + g_3)^{2d-1} = x_3^{2d-1}$ by comparing the coefficients of x_1^d. This implies that $g_3 = 0$. Hence, we have

$$\phi(x_3) = z_3 = \alpha_3 x_3 = \phi_{\alpha_3}(x_3). \tag{4.4}$$

From this, we know that ϕ commutes with the substitution $x_3 \mapsto 0$. By (1.2), we see that y_1 is sent to $x_1 - \theta'(x_2)\theta(x_2)$ by the substitution $x_3 \mapsto 0$. Since $y_1 = \phi(y_1)$, it follows that

$$x_1 - \theta'(x_2)\theta(x_2) = \phi(x_1) - \phi(\theta'(x_2)\theta(x_2)) = \alpha_1 x_1 + (\text{an element of } k[x_2, x_3])$$

by (4.3) and (4.2). This gives that $\alpha_1 = 1$. Therefore, we conclude that

$$\phi(x_1) = x_1 + g_1.$$

Since $y_3 = x_3$, we have $\phi(y_3) = \alpha_3 y_3$ by (4.4). Hence, we get

$$k[y_1, y_2, y_3] = k[\phi(y_1), y_2, \phi(y_3)].$$

By Lemma 4.1, this implies that

$$\phi(y_2) = \beta_2 y_2 + h$$

for some $\beta_2 \in k^\times$ and $h \in k[y_1, y_3] = k[y_1, x_3]$. We note that $\deg_{\mathbf{v}} h \le 3\mathbf{e}_1$, since $\deg_{\mathbf{v}} \phi(y_2) = 3\mathbf{e}_1$ by Proposition 3.1 (i), and $\deg_{\mathbf{v}} y_2 = \deg_{\mathbf{v}} x_1 x_3^2 = 3\mathbf{e}_1$ by (3.4). We prove that h belongs to $k[x_3]$. Suppose to the contrary that h does not belong to $k[x_3]$. Then, we have $\deg_{\mathbf{v}}^S h \ge \deg_{\mathbf{v}} y_1$, where $S := \{y_1, x_3\}$. Since $y_1^{\mathbf{v}} \approx x_1^d x_3^{2d-1}$ and x_3 are algebraically independent over k, we have $\deg_{\mathbf{v}} h = \deg_{\mathbf{v}}^S h$. Hence, we get $\deg_{\mathbf{v}} h \ge \deg_{\mathbf{v}} y_1 = (3d-1)\mathbf{e}_1 > 3\mathbf{e}_1$, a contradiction. This proves that h belongs to $k[x_3]$. From (1.1), we see that

$$\begin{aligned}
0 &= \alpha_3 x_3\big(y_1 - \phi(y_1)\big) \\
&= \alpha_3\big(x_1 x_3 + \theta(x_2) - \theta(y_2)\big) - \big(\alpha_3(x_1 + g_1)x_3 + \theta(\alpha_2 x_2 + g_2) - \theta(\beta_2 y_2 + h)\big) \\
&= \big(\theta(\beta_2 y_2 + h) - \alpha_3\theta(y_2)\big) - \big(\alpha_3 g_1 x_3 + \theta(\alpha_2 x_2 + g_2) - \alpha_3\theta(x_2)\big).
\end{aligned}$$

Hence, we have

$$p := \theta(\beta_2 y_2 + h) - \alpha_3\theta(y_2) = \alpha_3 g_1 x_3 + \theta(\alpha_2 x_2 + g_2) - \alpha_3\theta(x_2). \tag{4.5}$$

Since h belongs to $k[x_3] = k[y_3]$, we see that p belongs to $k[y_2, y_3]$. As an element of $k[y_2, y_3]$, we may consider the partial derivative $\partial p/\partial y_i$ for $i = 2, 3$. Then, we have

$$\frac{\partial p}{\partial x_1} = \frac{\partial y_2}{\partial x_1}\frac{\partial p}{\partial y_2} + \frac{\partial y_3}{\partial x_1}\frac{\partial p}{\partial y_3} = x_3^2\frac{\partial p}{\partial y_2}$$

by chain rule. Since g_1 and g_2 are elements of $k[x_2, x_3]$, we know that p belongs to $k[x_2, x_3]$ by (4.5). Hence, we have $\partial p/\partial x_1 = 0$. Thus, we get

$$0 = \frac{\partial p}{\partial y_2} = \beta_2\theta'(\beta_2 y_2 + h) - \alpha_3\theta'(y_2)$$

by the preceding equality. This implies that h is algebraic over $k(y_2)$. Since h is an element of $k[y_3]$, it follows that h belongs to k. Consequently, p belongs to $k[y_2]$. Since $\partial p/\partial y_2 = 0$, we conclude that p belongs to k.

We prove that $\beta_2 \neq 1$. Suppose to the contrary that $\beta_2 = 1$. Then, the coefficient of y_2^d in p is equal to $c(1 - \alpha_3)$. Since p belongs to k, it follows that $\alpha_3 = 1$. Hence, we get $\phi(y_3) = y_3$ by (4.4). Since $\beta_2 = \alpha_3 = 1$, we have

$$p = \theta(y_2 + h) - \theta(y_2) = \sum_{i=1}^{d} \frac{\theta^{(i)}(y_2)}{i!} h^i.$$

Since p belongs to k, this implies that $h = 0$. Thus, we get $\phi(y_2) = \beta_2 y_2 + h = y_2$. Since $\phi(y_1) = y_1$, we conclude that $\phi = \mathrm{id}_{k[\mathbf{x}]}$, a contradiction. This proves that $\beta_2 \neq 1$. Set $\kappa' = h/(1 - \beta_2)$, and write

$$\theta(z) = \sum_{i=0}^{d} u_i'(z - \kappa')^i,$$

where $u_i' \in k$ for $i = 0, \ldots, d$. Then, we have

$$\phi(y_2 - \kappa') = (\beta_2 y_2 + h) - \frac{h}{1 - \beta_2} = \beta_2\left(y_2 - \frac{h}{1 - \beta_2}\right) = \beta_2(y_2 - \kappa').$$

Hence, we know that

$$\begin{aligned} p &= \theta(\beta_2 y_2 + h) - \alpha_3\theta(y_2) = \theta(\phi(y_2)) - \alpha_3\theta(y_2) \\ &= \sum_{i=0}^{d} u_i'\phi(y_2 - \kappa')^i - \alpha_3\sum_{i=0}^{d} u_i'(y_2 - \kappa')^i = \sum_{i=0}^{d} u_i'(\beta_2^i - \alpha_3)(y_2 - \kappa')^i. \end{aligned}$$

Since p belongs to k, it follows that $p = u_0'(1 - \alpha_3)$, and $u_i' = 0$ or $\beta_2^i = \alpha_3$ for $i = 1, \ldots, d$.

We show that $\kappa' = \kappa$ by contradiction. Suppose that $\kappa'' := \kappa' - \kappa$ is nonzero. Since $u_{d-1} = 0$ by the definition of κ, we may write

$$\theta(z) = u_d(z - \kappa' + \kappa'')^d + \sum_{i=0}^{d-2} u_i(z - \kappa)^i = u_d(z - \kappa')^d + d\kappa'' u_d(z - \kappa')^{d-1} + \cdots.$$

Hence, we get $u_d' = u_d$ and $u_{d-1}' = d\kappa'' u_d$. Since $u_d \neq 0$, and $\kappa'' \neq 0$ by supposition, we know that u_d' and u_{d-1}' are nonzero. Hence, we have $\beta_2^i = \alpha_3$

for $i = d, d-1$. Thus, we get $\beta_2 = 1$, a contradiction. This proves that $\kappa' = \kappa$. Therefore, we have $u_i' = u_i$ for $i = 0, \dots, d$. Consequently, we get

$$p = u_0'(1-\alpha_3) = u_0(1-\alpha_3) = (1-\alpha_3)\theta(\kappa). \tag{4.6}$$

To complete the proof, we verify that $\phi(x_i) = \phi_{\alpha_3}(x_i)$ for $i = 1, 2$. Since $\sigma_\theta(f_\theta) = f_\theta$, $\phi(y_1) = y_1$ and $\phi(y_3) = \alpha_3 y_3$, we have

$$\begin{aligned}\phi(f_\theta) &= \phi(\sigma_\theta(f_\theta)) = \phi\big(y_1y_3 + \theta(y_2)\big) = y_1(\alpha_3 y_3) + \theta(\phi(y_2)) \\ &= \alpha_3 f_\theta - \alpha_3\theta(y_2) + \theta(\phi(y_2)) = \alpha_3 f_\theta + p.\end{aligned}$$

Hence, we get

$$\phi(y_2) = \phi(x_2 + f_\theta x_3) = (\alpha_2 x_2 + g_2) + (\alpha_3 f_\theta + p)(\alpha_3 x_3).$$

On the other hand, we have

$$\phi(y_2) = \beta_2 y_2 + h = \beta_2(x_2 + f_\theta x_3) + h.$$

Since g_2 and h are elements of $k[x_3]$ and k, respectively, the monomial x_2 does not appear in g_2 and h. Clearly, the monomial x_2 does not appear in a multiple of x_3. Thus, we get $\alpha_2 = \beta_2$ by comparing the coefficients of x_2. Equating the two expressions of $\phi(y_2)$, we obtain

$$g_2 - h = \big(\beta_2 f_\theta - \alpha_3(\alpha_3 f_\theta + p)\big)x_3 = \big((\alpha_2 - \alpha_3^2)f_\theta - \alpha_3 p\big)x_3.$$

Note that $g_2 - h$ belongs to $k[x_3]$, while f_θ does not belong to $k[x_3]$. Since p is a constant, it follows that $\alpha_2 = \alpha_3^2$ and $g_2 = h - \alpha_3 p x_3$. Therefore, we have

$$\begin{aligned}\phi(x_2 - \kappa) &= \phi(x_2 - \kappa') = \alpha_2 x_2 + g_2 - \frac{h}{1-\beta_2} = \alpha_2 x_2 + (h - \alpha_3 p x_3) - \frac{h}{1-\alpha_2} \\ &= \alpha_2\left(x_2 - \frac{h}{1-\alpha_2}\right) - \alpha_3 p x_3 = \alpha_3^2(x_2 - \kappa) + \alpha_3(\alpha_3 - 1)\theta(\kappa)x_3\end{aligned}$$

because of (4.6). This proves that $\phi(x_2) = \phi_{\alpha_3}(x_2)$. By (4.5) and (4.6), we have

$$g_1 = \frac{\alpha_3\theta(x_2) - \theta(\alpha_2 x_2 + g_2) + p}{\alpha_3 x_3} = \frac{\alpha_3\theta(x_2) - \phi(\theta(x_2)) + (1-\alpha_3)\theta(\kappa)}{\alpha_3 x_3} = g_{\alpha_3}.$$

Since $\phi(x_1) = x_1 + g_1$, this proves that $\phi(x_1) = \phi_{\alpha_3}(x_1)$. Therefore, we conclude that $\phi = \phi_{\alpha_3}$. Since $g_{\alpha_3} = g_1$ belongs to $k[\mathbf{x}]$, we know that α_3 belongs to T_θ as noted before Theorem 1.1. This completes the proof of Theorem 1.5 (ii), and thereby completing the proof of Theorem 1.1.

CHAPTER 7

Locally nilpotent derivations of rank three

1. Main result

Recall that the *rank* of $D \in \mathrm{Der}_k\, k[\mathbf{x}]$ is defined to be the minimal number $r \geq 0$ for which there exists $\sigma \in \mathrm{Aut}(k[\mathbf{x}]/k)$ such that $D(\sigma(x_i)) \neq 0$ for $i = 1, \ldots, r$. It is difficult to handle $D \in \mathrm{LND}_k\, k[\mathbf{x}]$ with $\mathrm{rank}\, D = 3$.

In this chapter, we construct families of elements of $\mathrm{LND}_k\, k[\mathbf{x}]$ using "local slice construction" due to Freudenburg [**7**]. Then, we prove that most of the members of the families are of rank three, for which the exponential automorphisms are wild.

For $i = 0, 1$, take $t_i \in \mathbf{N}$ and $\alpha_j^i \in k$ for $j = 1, \ldots, t_i - 1$. First, we define a sequence $(b_i)_{i=0}^{\infty}$ of integers by

$$b_0 = b_1 = 0 \quad \text{and} \quad b_{i+1} = t_i b_i - b_{i-1} + \xi_i \quad \text{for} \quad i \geq 1,$$

where $t_i := t_0$ if i is an even number, and $t_i := t_1$ otherwise, and where $\xi_i := 1$ if $i \equiv 0, 1 \pmod 4$, and $\xi_i := -1$ otherwise. Next, for each $i \geq 0$, we define a polynomial $\eta_i(y, z) \in k[y, z]$ by

$$\eta_i(y, z) = z^{t_i b_i + 1} + \sum_{j=1}^{t_i - 1} \alpha_j^i y^j z^{(t_i - j) b_i} + y^{t_i} \qquad \text{if } i \equiv 0, 1 \pmod 4$$

$$\eta_i(y, z) = y^{t_i} + \sum_{j=1}^{t_i - 1} \alpha_j^i z^{j b_i - 1} y^{t_i - j} + z^{t_i b_i - 1} \qquad \text{otherwise,}$$

where $\alpha_j^i := \alpha_j^0$ if i is an even number, and $\alpha_j^i := \alpha_j^1$ otherwise. Set

$$r = x_1 x_2 x_3 - \sum_{i=1}^{t_0} \alpha_i^0 x_2^i - \sum_{j=1}^{t_1} \alpha_j^1 x_1^j,$$

where $\alpha_{t_0}^0 := \alpha_{t_1}^1 := 1$. Then, we define a sequence $(f_i)_{i=0}^{\infty}$ of rational functions by

$$f_0 = x_2, \quad f_1 = x_1 \quad \text{and} \quad f_{i+1} = \eta_i(f_i, r) f_{i-1}^{-1} \quad \text{for} \quad i \geq 1.$$

Set $q_i = \eta_i(f_i, r)$ for each $i \geq 1$. Then, we have

$$q_1 = \eta_1(x_1, r) = r + \sum_{j=1}^{t_1} \alpha_j^1 x_1^j = x_1 x_2 x_3 - \sum_{i=1}^{t_0} \alpha_i^0 x_2^i. \tag{1.1}$$

Hence, we get

$$f_2 = q_1 x_2^{-1} = x_1 x_3 - \theta(x_2), \quad r = x_2 f_2 - \sum_{j=1}^{t_1} \alpha_j^1 x_1^j, \tag{1.2}$$

where $\theta(z) := \sum_{i=1}^{t_0} \alpha_i^0 z^{i-1}$. When $t_0 = 2$, we have $f_2 = x_1x_3 - x_2 - \alpha_1^0$. In this case, we can define $\tau_2 \in \mathrm{T}(k, \mathbf{x})$ by

$$\tau_2(x_1) = f_2, \quad \tau_2(x_2) = x_1 \quad \text{and} \quad \tau_2(x_3) = x_3. \tag{1.3}$$

For each $g_1, g_2 \in k[\mathbf{x}]$, we define $\Delta_{(g_1,g_2)} \in \mathrm{Der}_k\, k[\mathbf{x}]$ by

$$dg_1 \wedge dg_2 \wedge dg = \Delta_{(g_1,g_2)}(g) dx_1 \wedge dx_2 \wedge dx_3$$

for $g \in k[\mathbf{x}]$. Then, $k[g_1, g_2]$ is always contained in $\ker \Delta_{(g_1,g_2)}$. When f_i and f_{i+1} belong to $k[\mathbf{x}]$, we consider the derivation

$$D_i := \Delta_{(f_{i+1}, f_i)}$$

of $k[\mathbf{x}]$. For example, since $f_0 = x_2$ and $f_1 = x_1$ by definition, $D_0 = \Delta_{(x_1,x_2)}$ is the partial derivation of $k[\mathbf{x}]$ in x_3. By (1.2), we see that $D_1 = \Delta_{(f_2,x_1)}$ is the triangular derivation of $k[\mathbf{x}]$ defined by

$$D_1(x_1) = 0, \;\; D_1(x_2) = x_1 \;\text{ and }\; D_1(x_3) = \theta'(x_2) = \sum_{i=2}^{t_0} (i-1)\alpha_i^0 x_2^{i-2}, \tag{1.4}$$

since

$$df_2 \wedge dx_1 \wedge dx_2 = \frac{\partial f_2}{\partial x_3} dx_1 \wedge dx_2 \wedge dx_3, \;\; df_2 \wedge dx_1 \wedge dx_3 = -\frac{\partial f_2}{\partial x_2} dx_1 \wedge dx_2 \wedge dx_3.$$

We say that $D \in \mathrm{Der}_k\, k[\mathbf{x}]$ is *irreducible* if $D(k[\mathbf{x}])$ is contained in no proper principal ideal of $k[\mathbf{x}]$, or equivalently $D(x_1)$, $D(x_2)$ and $D(x_3)$ have no common factor. Here, "no common factor" means "no non-constant common factor". Then, D_0 is clearly irreducible, and D_1 is irreducible if and only if $t_0 \geq 2$. We mention that, if $t_0 = 1$, then $\ker D_1 = k[x_1, x_3]$ is not equal to $k[f_1, f_2] = k[x_1, x_1x_3 - 1]$.

We can similarly construct the sequence of rational functions from the same data

$$t_0, \quad t_1, \quad (\alpha_j^0)_{j=1}^{t_0-1} \quad \text{and} \quad (\alpha_j^1)_{j=1}^{t_1-1} \tag{1.5}$$

by interchanging the roles of t_0 and t_1, and $(\alpha_j^0)_{j=1}^{t_0-1}$ and $(\alpha_j^1)_{j=1}^{t_1-1}$, respectively. To distinguish it from the original one, we denote it by $(f_i')_{i=0}^{\infty}$. If $t_1 = 2$, then we can define $\tau_2' \in \mathrm{T}(k, \mathbf{x})$ by $\tau_2'(x_1) = f_2'$, $\tau_2'(x_2) = x_1$ and $\tau_2'(x_3) = x_3$. When f_i' and f_{i+1}' belong to $k[\mathbf{x}]$, we define $D_i' = \Delta_{(f_{i+1}', f_i')}$.

Let I be the set of $i \in \mathbf{N}$ such that

$$a_j := t_j b_j + \xi_j > 0$$

for $j = 1, \ldots, i$. Then, we have

$$I = \begin{cases} \{1\} & \text{if } t_0 = 1 \\ \{1,2\} & \text{if } (t_0, t_1) = (2,1) \\ \{1,2,3,4\} & \text{if } (t_0, t_1) = (3,1) \\ \mathbf{N} & \text{otherwise} \end{cases} \tag{1.6}$$

as will be shown in Section 3. We note that $a_i = \xi_i = 1$ for $i = 0, 1$, since $b_0 = b_1 = 0$. For each $i \geq 1$, we have

$$a_{i+1} = t_{i+1} a_i - a_{i-1}, \tag{1.7}$$

since

$$\begin{aligned} a_{i+1} - t_{i+1}a_i + a_{i-1} &= (t_{i+1}b_{i+1} + \xi_{i+1}) - t_{i+1}(t_ib_i + \xi_i) + (t_{i-1}b_{i-1} + \xi_{i-1}) \\ &= t_{i+1}(b_{i+1} - t_ib_i + b_{i-1} - \xi_i) + \xi_{i+1} + \xi_{i-1} = 0. \end{aligned}$$

Now, we are ready to state the main results of this chapter.

Theorem 1.1. *In the notation above, the following assertions hold for each* $i \in I$:
(i) f_i *and* f_{i+1} *belong to* $k[\mathbf{x}]$, D_i *belongs to* $\mathrm{LND}_k\, k[\mathbf{x}]$ *and* $D_i(r) = f_if_{i+1}$. *Moreover, we have the following*:
(a) *If* i *is the maximum of* I, *then* D_i *is not irreducible and* $\ker D_i \neq k[f_i, f_{i+1}]$.
(b) *If* i *is not the maximum of* I, *then* D_i *is irreducible and* $\ker D_i = k[f_i, f_{i+1}]$.
(ii) *If* $t_0 = 2$, *then we have* $\tau_2^{-1} \circ D_i \circ \tau_2 = D'_{i-1}$. *Hence,* D_2 *is tamely triangularizable. If* $t_0 = t_1 = 2$, *then we have* $\tau^{-1} \circ D_i \circ \tau = D_0$, *where*

$$\tau := \begin{cases} (\tau_2 \circ \tau_2')^{i/2} & \text{if } i \text{ is an even number} \\ (\tau_2 \circ \tau_2')^{(i-1)/2} \circ \tau_2 & \text{otherwise.} \end{cases}$$

(iii) *If* $t_0 = 2$, $t_1 \geq 3$ *and* $i \geq 3$, *or if* $t_0 \geq 3$ *and* $i \geq 2$, *then* $\exp hD_i$ *is wild for each* $h \in \ker D_i \setminus \{0\}$.

By means of this theorem, we can completely determine when $\exp hD_i$ is tame or wild for $h \in \ker D_i \setminus \{0\}$ and $i \in I$ as follows. First, consider the case of $i = 1$. Since D_1 is triangular and $D_1(x_1) = 0$ by (1.4), we see that hD_1 is triangular if h belongs to $k[x_1]$. In this case, $\exp hD_1$ is tame. Assume that h does not belong to $k[x_1]$. If $t_0 = 1$, then we have $D_1(x_3) = 0$. Hence, $\exp hD_1$ is elementary, and so tame. When $t_0 \geq 2$, we have $D_1(x_j) \neq 0$ for $j = 2, 3$. By Theorem 2.3, we know that $\exp hD_1$ is tame if and only if

$$\frac{\partial D_1(x_3)}{\partial x_2} = \sum_{i=3}^{t_0} (i-1)(i-2)\alpha_i^0 x_2^{i-3} \quad \text{belongs to} \quad D(x_2)k[x_1, x_2] = x_1k[x_1, x_2],$$

and hence if and only if $t_0 = 2$. Therefore, $\exp hD_1$ is tame if and only if h belongs to $k[x_1]$ or $t_0 \leq 2$. When $t_0 = 1$, we have $I = \{1\}$. Hence, this case is completed. Assume that $t_0 = 2$. Then, we have

$$hD_2 = \tau_2 \circ (\tau_2^{-1}(h)D'_1) \circ \tau_2^{-1}$$

by (ii). Since τ_2 is tame, $\exp hD_2$ is tame if and only if so is $\exp \tau_2^{-1}(h)D'_1$. From the preceding discussion, it follows that $\exp \tau_2^{-1}(h)D'_1$ is tame if and only if $\tau_2^{-1}(h)$ belongs to $k[x_1]$ or $t_1 \leq 2$. Since $\tau_2(x_1) = f_2$ by definition, $\tau_2^{-1}(h)$ belongs to $k[x_1]$ if and only if h belongs to $k[f_2]$. Therefore, we conclude that $\exp hD_2$ is tame if and only if h belongs to $k[f_2]$ or $t_1 \leq 2$ when $t_0 = 2$. If $(t_0, t_1) = (2, 1)$, then we have $I = \{1, 2\}$. Hence, this case is completed. If $t_0 = t_1 = 2$, then we have $hD_i = \tau \circ (\tau^{-1}(h)D_0) \circ \tau^{-1}$ for each $i \in I$ by (ii). Since $D_0(x_j) = 0$ for $j = 1, 2$, we see that $\exp \tau^{-1}(h)D_0$ is elementary. Hence, it follows that $\exp hD_i$ is tame by the tameness of τ. If $t_0 = 2$, $t_1 \geq 3$ and $i \geq 3$, or if $t_0 \geq 3$ and $i \geq 2$, then $\exp hD_i$ is wild due to (iii). Therefore, we have completely determined when $\exp hD_i$ is tame or wild for $h \in \ker D_i \setminus \{0\}$ and $i \in I$.

By the discussion above, we see that $\exp hD_i$ is tame only if $i = 1$ or $(i, t_0) = (2, 2)$ or $(t_0, t_1) = (2, 2)$. In this case, D_i is tamely triangularizable, and hence kills a tame coordinate of $k[\mathbf{x}]$ over k. Thanks to Theorem 1.3, this implies that hD_i is tamely triangularizable if $\exp hD_i$ is tame. Clearly, $\exp hD_i$ is tame if hD_i is tamely triangularizable. Therefore, we get the following corollary to Theorem 1.1.

Corollary 1.2. *For $h \in \ker D_i \setminus \{0\}$ and $i \in I$, it holds that $\exp hD_i$ is tame if and only if hD_i is tamely triangularizable.*

If D is a derivation of $k[\mathbf{x}]$, then

$$\operatorname{pl} D := D(k[\mathbf{x}]) \cap \ker D$$

forms an ideal of $\ker D$, and is called the *plinth ideal* of D. Assume that D is locally nilpotent. Then, $\operatorname{pl} D$ is not a zero ideal unless $D = 0$. Moreover, $\operatorname{pl} D$ is always a principal ideal of $\ker D$ by Daigle-Kaliman [**5**, Theorem 1]. Hence, the notation $\operatorname{pl} D = (p)$ will mean that $\operatorname{pl} D = p \ker D$. The plinth ideals are important in the study of the ranks of locally nilpotent derivations.

In the following theorem, we completely determine $\operatorname{pl} D_i$ and $\operatorname{rank} D_i$ for $i \in I$.

Theorem 1.3. *The following assertions hold for each $i \in I$:*
(i) *If $t_0 = 1$ or $(t_0, t_1, i) = (2, 1, 2)$, then we have $\operatorname{pl} D_i = (f_i)$ and $\operatorname{rank} D_i = 1$.*
(ii) *If $(t_0, i) = (2, 1)$ or $t_0 = t_1 = 2$, then we have $\operatorname{pl} D_i = \ker D_i$ and $\operatorname{rank} D_i = 1$.*
(iii) *If $t_0 = 2$, $t_1 \geq 3$ and $i = 2$, or if $t_0 \geq 3$ and $i = 1$, then we have $\operatorname{pl} D_i = (f_i)$ and $\operatorname{rank} D_i = 2$.*
(iv) *If $t_0 = 2$, $t_1 \geq 3$ and $i \geq 3$, or if $t_0 \geq 3$, $(t_0, t_1) \neq (3, 1)$ and $i \geq 2$, then we have $\operatorname{pl} D_i = (f_i f_{i+1})$ and $\operatorname{rank} D_i = 3$.*
(v) *If $(t_0, t_1) = (3, 1)$, then we have $\operatorname{pl} D_2 = (f_3)$ and $\operatorname{rank} D_2 = 2$, $\operatorname{pl} D_3 = \ker D_3$ and $\operatorname{rank} D_3 = 1$, and $\operatorname{pl} D_4 = (f_4)$ and $\operatorname{rank} D_4 = 1$.*

On the homogeneity of f_i's, we have the following result. Assume that $\alpha_j^i = 0$ for $i = 0, 1$ and $j = 1, \ldots, t_i - 1$. Then, we have $r = x_1 x_2 x_3 - x_2^{t_0} - x_1^{t_1}$ and $f_2 = x_1 x_3 - x_2^{t_0 - 1}$. Put

$$\mathbf{t} := (t_0, t_1, t_0 t_1 - t_0 - t_1).$$

Then, it is easy to check that r and f_2 are $\mathbf{t}$-homogeneous. Moreover, we have the following proposition.

Proposition 1.4. *If $\alpha_j^i = 0$ for $i = 0, 1$ and $j = 1, \ldots, t_i - 1$, then f_i and f_{i+1} are $\mathbf{t}$-homogeneous for each $i \in I$.*

Next, we construct another family of elements of $\operatorname{LND}_k k[\mathbf{x}]$ by making use of $(f_i)_{i=0}^{\infty}$. Take any

$$\lambda(y) \in k[y] \setminus \{0\} \quad \text{and} \quad \mu(y, z) = \sum_{j \geq 1} \mu_j(y) z^j \in z k[y, z],$$

where $\mu_j(y) \in k[y]$ for each j. For $i \geq 2$, we set

$$r_i = \lambda(f_i)\tilde{r} - \mu(f_i, f_{i-1}), \quad \text{where} \quad \tilde{r} := \begin{cases} x_2 & \text{if} \quad i = 2 \\ r & \text{if} \quad i \geq 3. \end{cases}$$

Then, we define

$$\tilde{f}_{i+1} = \tilde{\eta}_i\left(f_i, r_i\lambda(f_i)^{-1}\right) f_{i-1}^{-1}\lambda(f_i)^{a_i}, \quad \text{where } \tilde{\eta}_i(y,z) := \begin{cases} y+\theta(z) & \text{if } i=2 \\ \eta_i(y,z) & \text{if } i \geq 3. \end{cases}$$

When $\tilde{f}_{i+1}$ belongs to $k[\mathbf{x}]$, we consider the derivation $\tilde{D}_i := \Delta_{(\tilde{f}_{i+1},f_i)}$ of $k[\mathbf{x}]$.

To study $\tilde{D}_i$, we may assume that $\lambda(y)$ and $\mu(y,z)$ have no common factor for the following reason. Let $\nu(y)$ be a common factor of $\lambda(y)$ and $\mu(y,z)$, and let

$$\lambda_0(y) := \lambda(y)\nu(y)^{-1} \quad \text{and} \quad \mu_0(y,z) := \mu(y,z)\nu(y)^{-1}.$$

Then, we have

$$\begin{aligned} g :&= \tilde{\eta}_i\Big(f_i, \big(\lambda_0(f_i)\tilde{r} - \mu_0(f_i,f_{i-1})\big)\lambda_0(f_i)^{-1}\Big)\lambda_0(f_i)^{a_i} f_{i-1}^{-1} \\ &= \tilde{\eta}_i\left(f_i, r_i\lambda(f_i)^{-1}\right)\lambda(f_i)^{a_i} f_{i-1}^{-1}\nu(f_i)^{-a_i} = \nu(f_i)^{-a_i}\tilde{f}_{i+1}. \end{aligned}$$

Hence, we get $\tilde{D}_i = \Delta_{(\tilde{f}_{i+1},f_i)} = \nu(f_i)^{a_i}\Delta_{(g,f_i)}$. Therefore, the study of $\tilde{D}_i$ is reduced to the study of $\Delta_{(g,f_i)}$.

If $\mu(y,z)=0$, then we have $r_i\lambda(f_i)^{-1} = \tilde{r}$. In this case, we cannot obtain a new derivation as $\tilde{D}_i$ for the following reason. Since $\mu(y,z)=0$, we may assume that $\lambda(y)=c$ for some $c \in k^\times$ by the preceding discussion. Then, we have

$$\begin{aligned} \tilde{f}_3 &= \tilde{\eta}_2(f_2, c^{-1}r_2)f_1^{-1}c^{a_2} = \tilde{\eta}_2(f_2,x_2)f_1^{-1}c^{a_2} \\ &= c^{a_2}\big(f_2+\theta(x_2)\big)f_1^{-1} = c^{a_2}x_1x_3x_1^{-1} = c^{a_2}x_3 \end{aligned} \tag{1.8}$$

when $i=2$. This gives that $\tilde{D}_2 = c^{a_2}\Delta_{(x_3,f_2)}$. Since f_2 is a symmetric polynomial in x_1 and x_3 over $k[x_2]$, we may regard $\tilde{D}_2$ as $c^{a_2}\Delta_{(x_1,f_2)} = -c^{a_2}D_1$ by interchanging x_1 and x_3. When $i \geq 3$, we have $\tilde{r} = r$ and $\tilde{\eta}_i(y,z) = \eta_i(y,z)$ by definition. Since $r_i\lambda(f_i)^{-1} = \tilde{r}$ and $\lambda(y)=c$, it follows that $\tilde{f}_{i+1} = c^{a_i}f_{i+1}$. Thus, we get $\tilde{D}_i = c^{a_i}D_i$. Therefore, we may assume that $\mu(y,z) \neq 0$.

In the notation above, we have the following theorem.

Theorem 1.5. *Assume that $t_0 \geq 3$ and $i=2$, or $t_0 \geq 3$, $(t_0,t_1) \neq (3,1)$ and $i \geq 3$. If $\lambda(y) \in k[y]\setminus\{0\}$ and $\mu(y,z) \in zk[y,z]\setminus\{0\}$ have no common factor, then the following assertions hold*:
(i) *$\tilde{f}_{i+1}$ belongs to $k[\mathbf{x}]$, and $\tilde{D}_i$ is irreducible and locally nilpotent. Moreover, we have $\ker\tilde{D}_i = k[f_i,\tilde{f}_{i+1}]$, and*

$$\tilde{D}_i(r_i) = \begin{cases} \lambda(f_2)\tilde{f}_3 & \text{if } i=2 \\ \lambda(f_i)f_i\tilde{f}_{i+1} & \text{if } i \geq 3. \end{cases} \tag{1.9}$$

(ii) *For $h \in \ker\tilde{D}_i\setminus\{0\}$, it holds that $\exp h\tilde{D}_i$ is tame if and only if $i=2$, $\lambda(y)$ belongs to $k^\times$, $\mu(y,z)$ belongs to $zk[z]\setminus\{0\}$, and h belongs to $k[\tilde{f}_3]\setminus\{0\}$. If this is the case, then $h\tilde{D}_i$ is tamely triangularizable.*

In the statement (iii) of the following theorem, $\sqrt{(\lambda(y))}$ denotes the radical of the ideal $(\lambda(y))$ of $k[y]$.

Theorem 1.6. *Under the assumption of Theorem* 1.5*, the following assertions hold*:
(i) *If* $i \geq 3$*, then we have* $\operatorname{rank} \tilde{D}_i = 3$*.*
(ii) *If* $\lambda(y)$ *belongs to* $k^\times$*, then we have* $\operatorname{rank} \tilde{D}_2 = 2$ *and* $\operatorname{pl} \tilde{D}_2 = (\tilde{f}_3)$*. Moreover,* $\tilde{f}_3$ *is a coordinate of* $k[\mathbf{x}]$ *over* k*.*
(iii) *Assume that* $\lambda(y)$ *does not belong to* $k^\times$*. If* $t_0 \geq 4$*, or* $t_0 = 3$ *and* $\mu_j(y)$ *does not belong to* $\sqrt{(\lambda(y))}$ *for some* $j \geq 2$*, then we have* $\operatorname{rank} \tilde{D}_2 = 3$*. If* $t_0 = 3$ *and* $\mu_j(y)$ *belongs to* $\sqrt{(\lambda(y))}$ *for every* $j \geq 2$*, then we have* $\operatorname{pl} \tilde{D}_2 = (\tilde{f}_3)$*.*

In the last case of (iii), we do not know the rank of $\tilde{D}_2$ in general. However, if $\tilde{f}_3$ is a coordinate of $k[\mathbf{x}]$ over k, then we may conclude that $\operatorname{rank} \tilde{D}_2 = 2$ as follows. Since $\tilde{D}_2$ kills $\tilde{f}_3$, we have $1 \leq \operatorname{rank} \tilde{D}_2 \leq 2$ by the assumption that $\tilde{f}_3$ is a coordinate of $k[\mathbf{x}]$ over k. Suppose that $\operatorname{rank} \tilde{D}_2 = 1$. Then, we have $\ker \tilde{D}_2 = \sigma(k[x_2, x_3])$ for some $\sigma \in \operatorname{Aut}(k[\mathbf{x}]/k)$. Since $\ker \tilde{D}_2 = k[f_2, \tilde{f}_3]$ by Theorem 1.5 (i), it follows that

$$k[f_2, \tilde{f}_3][\sigma(x_1)] = k[\sigma(x_2), \sigma(x_3)][\sigma(x_1)] = \sigma(k[\mathbf{x}]) = k[\mathbf{x}].$$

This implies that f_2 is a coordinate of $k[\mathbf{x}]$ over k. Since $t_0 = 3$, we have $f_2 = x_1x_3 - (\alpha_1^0 + \alpha_2^0 x_2 + x_2^2)$. Hence, f_2 is changed to a polynomial with no linear monomial by the substitution $x_2 \mapsto x_2 - \alpha_2^0/2$. This implies that f_2 is not a coordinate of $k[\mathbf{x}]$ over k, a contradiction. Therefore, if $\tilde{f}_3$ is a coordinate of $k[\mathbf{x}]$ over k, then we have $\operatorname{rank} \tilde{D}_2 = 2$. Possibly, $\tilde{f}_3$ is always a coordinate of $k[\mathbf{x}]$ over k.

Finally, we discuss the locally nilpotent derivations of rank three given by Freudenburg [**7**, Section 4] (see also [**9**, Sections 5.5.2 and 5.5.3]). Assume that $t_0 = t_1 = 3$ and $\alpha_j^i = 0$ for $i = 0, 1$ and $j = 1, 2$. Then, we have $r = x_2 f_2 - x_1^3$ and $f_2 = x_1x_3 - x_2^2$ by (1.2). Moreover, we have $f_0 = x_2$ and $f_1 = x_1$, and

$$f_{i-1} f_{i+1} = f_i^3 + r^{a_i} \quad \text{for each} \quad i \geq 1$$

by the definition of $\eta_i(y, z)$'s. As mentioned, we have $a_0 = a_1 = 1$, and $a_{i+1} = 3a_i - a_{i-1}$ for $i \geq 1$ by (1.7). From these conditions, we know that $-D_i = \Delta_{(f_i, f_{i+1})}$ is the same as the locally nilpotent derivation of "Fibonacci type" by Freudenburg for each $i \geq 1$, where we regard x_1, x_2 and x_3 as x, $-y$ and z, respectively. It is previously known that $\Delta_{(f_i, f_{i+1})}$ has rank three if $i \geq 2$. In this case, $\exp h\Delta_{(f_i, f_{i+1})}$ is wild for each $h \in \ker \Delta_{(f_i, f_{i+1})} \setminus \{0\}$ by Theorem 1.1 (iii).

Next, let $\lambda(y) = y^l$ and $\mu(y, z) = -z^m$ for $l \geq 1$ and $m \geq 1$. Then, we have $r_2 = f_2^l x_2 + x_1^m$. Since $\tilde{\eta}_2(y, z) = y + z^2$ and $a_2 = t_2 a_1 - a_0 = 2$ by (1.7), it follows that

$$\begin{aligned} f_2 \tilde{f}_3 &= \tilde{\eta}_2\left(f_2, r_2 \lambda(f_2)^{-1}\right) \lambda(f_2)^{a_2} = \left(f_2 + (r_2 f_2^{-l})^2\right) f_2^{2l} = f_2^{2l+1} + r_2^2 \\ &= f_2^{2l}(x_1x_3 - x_2^2) + (f_2^l x_2 + x_1^m)^2 = x_1(f_2^{2l} x_3 + 2 f_2^l x_1^{m-1} x_2 + x_1^{2m-1}). \end{aligned}$$

Hence, we get

$$\tilde{f}_3 = f_2^{2l} x_3 + 2 f_2^l x_1^{m-1} x_2 + x_1^{2m-1}.$$

This shows that, if $m = 2l + 1$, then $-\tilde{D}_2 = \Delta_{(f_2, \tilde{f}_3)}$ is the same as the rank three homogeneous locally nilpotent derivation of "type $(2, 4l + 1)$" due to

Freudenburg, where we regard x_1, x_2 and x_3 as x, y and z, respectively. If $l \geq 1$ and $m \geq 1$, then $\exp h\Delta_{(f_2,\tilde{f}_3)}$ is wild for each $h \in \ker \Delta_{(f_2,\tilde{f}_3)} \setminus \{0\}$ by Theorem 1.5 (iii), since $\lambda(y) = y^l$ does not belong to $k^\times$. If $m \geq 2$, then we have $\operatorname{rank} \Delta_{(f_2,\tilde{f}_3)} = 3$ by Theorem 1.6 (iii), since $t_0 = 3$ and $\mu_m(y) = -1$ does not belong to $\sqrt{(\lambda(y))} = (y)$.

Note: Recently, Prof. Freudenburg kindly informed the author that he independently realized that Karaś-Zygadło [**13**, Theorem 2.1] implies the wildness of $\exp D$ for his homogeneous locally nilpotent derivation of type $(2,5)$ as follows: In this case, it holds that

$$\deg(\exp D)(x_1) = 9, \quad \deg(\exp D)(x_2) = 25, \quad \deg(\exp D)(x_3) = 41.$$

On the other hand, the above-mentioned result of Karaś-Zygadło implies that $\phi \in \mathrm{Aut}_{\mathbf{C}}\mathbf{C}[x_1,x_2,x_3]$ is wild if the following conditions hold:
(i) $\deg \phi(x_3) \geq \deg \phi(x_2) > \deg \phi(x_1) \geq 3$.
(ii) $\deg \phi(x_1)$ and $\deg \phi(x_2)$ are mutually prime odd numbers.
(iii) $\deg \phi(x_3)$ does not belong to $\mathbf{Z}_{\geq 0} \deg \phi(x_1) + \mathbf{Z}_{\geq 0} \deg \phi(x_2)$.

It is easy to check that $\exp D$ satisfies (i), (ii) and (iii). The author would like to thank Prof. Freudenburg for informing him the remark.

2. A reduction lemma

Recall that we defined elements τ_2 and τ_2' of $\mathrm{T}(k[x_3],\{x_1,x_2\})$ when $t_0 = 2$ and $t_1 = 2$, respectively. Let $k(\mathbf{x})$ be the field of fractions of $k[\mathbf{x}]$. Then, τ_2 and τ_2' uniquely extend to automorphisms of $k(\mathbf{x})$. By abuse of notation, we denote them by the same symbols τ_2 and τ_2'. The purpose of this section is to prove the following lemma, and Theorem 1.1 (ii) on the assumption that f_i and f_{i+1} belong to $k[\mathbf{x}]$ for each $i \in I$.

Lemma 2.1. (i) *If $t_0 = 2$, then we have $\tau_2(f_i') = f_{i+1}$ for each $i \geq 0$.*
(ii) *If $t_1 = 2$, then we have $\tau_2'(f_i) = f_{i+1}'$ for each $i \geq 0$.*
(iii) *If $t_0 = t_1 = 2$, then we have $(\tau_2 \circ \tau_2')^j(f_i) = f_{i+2j}$ for each $i, j \geq 0$.*

From the data (1.5), we can similarly define $(b_i)_{i=0}^\infty$, r, $(\eta_i(y,z))_{i=0}^\infty$, and $(a_i)_{i=0}^\infty$ by interchanging the roles of t_0 and t_1, and $(\alpha_j^0)_{j=1}^{t_0-1}$ and $(\alpha_j^1)_{j=1}^{t_1-1}$, respectively. To distinguish them from the original ones, we denote them by

$$(b_i')_{i=0}^\infty, \quad r', \quad (\eta_i'(y,z))_{i=0}^\infty \quad \text{and} \quad (a_i')_{i=0}^\infty,$$

respectively. We note that $b_0' = b_1' = 0$ and $b_{i+1}' = t_{i+1}b_i' - b_{i-1}' + \xi_i$ for each $i \geq 1$.

In the rest of this section, we always assume that $t_0 = 2$.

Lemma 2.2. *For each $i \geq 0$, we have*

$$b_i' = b_{i+1} + \frac{\xi_{i+1} - \xi_i}{2} = \begin{cases} b_{i+1} - 1 & \text{if } i \equiv 1 \pmod 4 \\ b_{i+1} + 1 & \text{if } i \equiv 3 \pmod 4 \\ b_{i+1} & \text{otherwise.} \end{cases}$$

Proof. We prove the first equality by induction on i. Since $b_0' = b_1' = 0$, and

$$b_1 + \frac{\xi_1 - \xi_0}{2} = \frac{1-1}{2} = 0 \quad \text{and} \quad b_2 + \frac{\xi_2 - \xi_1}{2} = 1 + \frac{(-1)-1}{2} = 0,$$

the equality holds for $i = 0, 1$. Assume that $i \geq 2$. Then, we have $b'_j = b_{j+1} + (\xi_{j+1} - \xi_j)/2$ for $j = i-2, i-1$ by induction assumption. Hence, we get

$$b'_i = t_i b'_{i-1} - b'_{i-2} + \xi_{i-1} = t_i \left(b_i + \frac{\xi_i - \xi_{i-1}}{2} \right) - \left(b_{i-1} + \frac{\xi_{i-1} - \xi_{i-2}}{2} \right) + \xi_{i-1}.$$

Since $t_i b_i - b_{i-1} = b_{i+1} - \xi_i$ and $-(\xi_{i-1} - \xi_{i-2}) = \xi_{i+1} - \xi_i$, the right-hand side is equal to

$$b_{i+1} - \xi_i + t_i \frac{\xi_i - \xi_{i-1}}{2} + \frac{\xi_{i+1} - \xi_i}{2} + \xi_{i-1} = b_{i+1} + (t_i - 2)\frac{\xi_i - \xi_{i-1}}{2} + \frac{\xi_{i+1} - \xi_i}{2}.$$

Since $t_i = 2$ if i is an even number, and $\xi_i - \xi_{i-1} = 0$ otherwise, the right-hand side is equal to $b_{i+1} + (\xi_{i+1} - \xi_i)/2$. This proves that the first equality holds for every $i \geq 0$. The second equality is readily verified. $\square$

Since $t_0 = 2$ by assumption, we know by Lemma 2.2 that $\eta'_i(y, z)$ is equal to

$$\begin{aligned}
&z^{t_1 b'_i + 1} + \sum_{j=1}^{t_1} \alpha_j^1 y^j z^{(t_1 - j) b'_i} = z^{t_1 b_{i+1} + 1} + \sum_{j=1}^{t_1} \alpha_j^1 y^j z^{(t_1 - j) b_{i+1}} && \text{if} \quad i \equiv 0 \pmod 4 \\
&z^{2b'_i + 1} + \alpha_1^0 y z^{b'_i} + y^2 = y^2 + \alpha_1^0 y z^{b_{i+1} - 1} + z^{2b_{i+1} - 1} && \text{if} \quad i \equiv 1 \pmod 4 \\
&y^{t_1} + \sum_{j=1}^{t_1} \alpha_j^1 y^{t_1 - j} z^{j b'_i - 1} = y^{t_1} + \sum_{j=1}^{t_1} \alpha_j^1 y^{t_1 - j} z^{j b_{i+1} - 1} && \text{if} \quad i \equiv 2 \pmod 4 \\
&y^2 + \alpha_1^0 y z^{b'_i - 1} + z^{2b'_i - 1} = z^{2b_{i+1} + 1} + \alpha_1^0 y z^{b_{i+1}} + y^2 && \text{if} \quad i \equiv 3 \pmod 4
\end{aligned}$$

for $i \geq 0$. In each case, the right-hand side is equal to $\eta_{i+1}(y, z)$. Hence, we get

$$\eta'_i(y, z) = \eta_{i+1}(y, z)$$

for $i \geq 0$. Since $a'_i = \deg_z \eta'_i(y, z)$ and $a_{i+1} = \deg_z \eta_{i+1}(y, z)$, this implies that $a'_i = a_{i+1}$ for each $i \geq 0$.

Now, let us prove Lemma 2.1. First, we prove (i) by induction on i. Since $f'_0 = x_2$ and $f'_1 = x_1$, we get

$$\tau_2(f'_0) = \tau_2(x_2) = x_1 = f_1 \quad \text{and} \quad \tau_2(f'_1) = \tau_2(x_1) = f_2$$

by (1.3). Hence, the statement holds for $i = 0, 1$. Assume that $i \geq 2$. Since $t_0 = 2$, we see that

$$r' = x_1 x_2 x_3 - \sum_{i=1}^{t_1} \alpha_i^1 x_2^i - \sum_{j=1}^{2} \alpha_j^0 x_1^j = x_1(x_2 x_3 - \alpha_1^0 - x_1) - \sum_{i=1}^{t_1} \alpha_i^1 x_2^i.$$

Hence, we get

$$\tau_2(r') = f_2(x_1 x_3 - \alpha_1^0 - f_2) - \sum_{i=1}^{t_1} \alpha_i^1 x_1^i = f_2 x_2 - \sum_{i=1}^{t_1} \alpha_i^1 x_1^i = r$$

in view of (1.2). Since $\tau_2(f'_j) = f_{j+1}$ for $j = i-2, i-1$ by induction assumption, and $\eta'_{i-1}(y, z) = \eta_i(y, z)$ as shown above, it follows that

$$\tau_2(f'_i) = \tau_2\left(\eta'_{i-1}(f'_{i-1}, r')(f'_{i-2})^{-1}\right) = \eta_i(f_i, r) f_{i-1}^{-1} = f_{i+1}.$$

This proves (i). We can prove (ii) similarly. To prove (iii), assume that $t_0 = t_1 = 2$. Then, we have $\tau_2(f_i') = f_{i+1}$ and $\tau_2'(f_i) = f_{i+1}'$ for each $i \geq 0$ by (i) and (ii). Hence, know that

$$(\tau_2 \circ \tau_2')(f_i) = \tau_2(f_{i+1}') = f_{i+2}.$$

Therefore, we get $(\tau_2 \circ \tau_2')^j(f_i) = f_{i+2j}$ for each $i, j \geq 0$. This proves (iii), and thus completes the proof of Lemma 2.1.

For $g_1, g_2, g_3 \in k[\mathbf{x}]$, define an endomorphism ψ of the k-algebra $k[\mathbf{x}]$ by $\psi(x_i) = g_i$ for $i = 1, 2, 3$. Then, we have $\Delta_{(g_1,g_2)}(g_3) = \det J\psi$. From this, it follows that

$$\phi^{-1} \circ \Delta_{(\phi(g_1),\phi(g_2))} \circ \phi = (\det J\phi)\Delta_{(g_1,g_2)}. \tag{2.1}$$

Actually, since $J(\phi \circ \psi) = \phi(J\psi) \cdot J\phi$, and $\det J\phi$ is a constant, we have

$$\begin{aligned}(\phi^{-1} \circ \Delta_{(\phi(g_1),\phi(g_2))} \circ \phi)(g_3) &= \phi^{-1}(\det J(\phi \circ \psi)) = \phi^{-1}\big(\det\big(\phi(J\psi) \cdot J\phi\big)\big) \\ &= \phi^{-1}(\phi(\det J\psi) \det J\phi) = (\det J\psi) \det J\phi = (\det J\phi)\Delta_{(g_1,g_2)}(g_3).\end{aligned}$$

Now, we prove Theorem 1.1 (ii) on the assumption that f_i and f_{i+1} belong to $k[\mathbf{x}]$ for each $i \in I$. Assume that $t_0 = 2$, and take any $i \in I$. Then, it follows that $f_{j-1}' = \tau_2^{-1}(f_j)$ belongs to $k[\mathbf{x}]$ for $j = i, i+1$ by Lemma 2.1 (i), since $i \geq 1$. Hence, we may consider D_{i-1}'. By definition, we see that $\det J\tau_2 = 1$. Thus, we know by (2.1) that

$$\tau_2^{-1} \circ D_i \circ \tau_2 = \tau_2^{-1} \circ \Delta_{(\tau_2(f_i'),\tau_2(f_{i-1}'))} \circ \tau_2 = (\det J\tau_2)\Delta_{(f_i',f_{i-1}')} = D_{i-1}'. \tag{2.2}$$

This proves the first part of Theorem 1.1 (ii). Since D_1' is triangular, and τ_2 is tame, the second part follows immediately.

To prove the last part, assume that $t_0 = t_1 = 2$, and define τ as in the theorem. Then, we know by Lemma 2.1 (iii) that

$$\tau(x_2) = (\tau_2 \circ \tau_2')^{i/2}(f_0) = f_i, \quad \tau(x_1) = (\tau_2 \circ \tau_2')^{i/2}(f_1) = f_{i+1}$$

if i is an even number. Since $\tau_2(f_0') = f_1$ and $\tau_2(f_1') = f_2$ by Lemma 2.1 (i), we have

$$\tau(x_2) = ((\tau_2 \circ \tau_2')^{(i-1)/2} \circ \tau_2)(f_0') = f_i, \quad \tau(x_1) = ((\tau_2 \circ \tau_2')^{(i-1)/2} \circ \tau_2)(f_1') = f_{i+1}$$

if i is an odd number. Hence, $\tau(x_2) = f_i$ and $\tau(x_1) = f_{i+1}$ hold in either case. Because $\det J\tau_2 = \det J\tau_2' = 1$, we get $\det J\tau = 1$. Thus, we conclude that

$$\tau^{-1} \circ D_i \circ \tau = \tau^{-1} \circ \Delta_{(\tau(x_1),\tau(x_2))} \circ \tau = (\det J\tau)\Delta_{(x_1,x_2)} = D_0$$

by the formula (2.1). This proves the last part of Theorem 1.1 (ii), and thereby completing the proof of Theorem 1.1 (ii).

3. Properties of $(a_i)_{i=0}^{\infty}$

Let us prove (1.6). By (1.7), we can check the first three cases immediately. Actually, we have $a_1 = 1$ and $a_2 = 0$ if $t_0 = 1$, $a_1 = a_2 = 1$ and $a_3 = 0$ if $(t_0, t_1) = (2, 1)$, and $a_1 = a_3 = a_4 = 1$, $a_2 = 2$ and $a_5 = 0$ if $(t_0, t_1) = (3, 1)$. If $t_0 = t_1 = 2$, then we have $a_{i+1} - a_i = a_i - a_{i-1}$ for each $i \geq 1$ by (1.7). Since $a_0 = a_1 = 1$, it follows that $a_i = 1$ for every $i \in \mathbf{N}$. Hence, we get $I = \mathbf{N}$.

By (ii) of the following lemma, we have $I = \mathbf{N}$ if $t_0 \geq 3$ and $(t_0, t_1) \neq (3, 1)$. Then, it follows that $I = \mathbf{N}$ if $t_0 = 2$ and $t_1 \geq 3$, since $a_{i+1} = a_i'$ for each $i \geq 0$ as mentioned after Lemma 2.2. Thus, (1.6) is proved in all cases.

Lemma 3.1. *If $t_0 \geq 3$ and $(t_0, t_1) \neq (3, 1)$, then the following assertions hold*:
(i) *For each $i \geq 1$, we have $a_{i+2} - a_i \geq 2$ if i is an even number, and $a_{i+2} - a_i \geq 1$ otherwise.*
(ii) *For each $i \geq 2$, we have $a_i \geq 2$ and $b_i \geq 1$.*
(iii) *For each $i \geq 3$, we have $a_i > t_i$.*
(iv) *For each $i \geq 3$, we have $b_i \geq 1$ if $i \equiv 0, 1 \pmod 4$, and $b_i \geq 2$ otherwise.*
(v) *For each $i \geq 2$ and $l \in \mathbf{N}$, we have $la_i \neq t_i$.*

To prove Lemma 3.1, we use the following lemma.

Lemma 3.2. *Let $\gamma_{i-2}, \dots, \gamma_{i+2}$ be five elements of an abelian group for some $i \in \mathbf{Z}$. Assume that there exists $l \in \mathbf{Z}$ such that $\gamma_{j+1} = t_{j+l}\gamma_j - \gamma_{j-1}$ for $j = i-1, i, i+1$. Then, we have $\gamma_{i+2} - \gamma_i = (t_0t_1 - 4)\gamma_i + (\gamma_i - \gamma_{i-2})$.*

Proof. Since $t_{j+l}\gamma_j = \gamma_{j+1} + \gamma_{j-1}$ for $j = i-1, i, i+1$, we get

$$\begin{aligned} t_{i+l-1}t_{i+l}\gamma_i &= t_{i+l-1}(\gamma_{i+1} + \gamma_{i-1}) \\ &= t_{(i+1)+l}\gamma_{i+1} + t_{(i-1)+l}\gamma_{i-1} = (\gamma_{i+2} + \gamma_i) + (\gamma_i + \gamma_{i-2}). \end{aligned}$$

This gives that $\gamma_{i+2} - \gamma_i = (t_0t_1 - 4)\gamma_i + (\gamma_i - \gamma_{i-2})$. ∎

Now, let us prove Lemma 3.1. To prove (i), we demonstrate that

$$a_{2i-1} \geq 0, \quad a_{2i} \geq 0, \quad a_{2i+1} - a_{2i-1} \geq 1, \quad a_{2i+2} - a_{2i} \geq 2$$

for each $i \geq 1$ by induction on i. Since $t_0 \geq 3$ and $(t_0, t_1) \neq (3, 1)$ by assumption, we have $t_0 \geq 4$ and $t_1 = 1$, or $t_0 \geq 3$ and $t_1 \geq 2$. Assume that $i = 1$. Then, we have $a_1 = 1$ and $a_2 = t_0 - 1 \geq 2$ by the definition of a_i. From (1.7), it follows that

$$\begin{aligned} a_3 - a_1 &= t_3a_2 - 2a_1 = t_1(t_0 - 1) - 2 \geq 1, \\ a_4 - a_2 &= t_4a_3 - 2a_2 = t_0(t_1(t_0 - 1) - 1) - 2(t_0 - 1) \\ &= t_0(t_1(t_0 - 1) - 3) + 2 \geq 2. \end{aligned}$$

Thus, the statement holds when $i = 1$. Assume that $i \geq 2$. Then, we have

$$a_{2i-3} \geq 0, \quad a_{2i-2} \geq 0, \quad a_{2i-1} - a_{2i-3} \geq 1, \quad a_{2i} - a_{2i-2} \geq 2$$

by induction assumption. This implies that $a_{2i-1} \geq 0$ and $a_{2i} \geq 0$. By (1.7), we know that $a_{j+1} = t_{j+1}a_j - a_{j-1}$ for $j = 2i-2, 2i-1, 2i$, since $j \geq 2i - 2 \geq 2$. Hence, we get

$$a_{2i+1} - a_{2i-1} = (t_0t_1 - 4)a_{2i-1} + (a_{2i-1} - a_{2i-3})$$

by Lemma 3.2. Since $t_0 \geq 4$ and $t_1 = 1$, or $t_0 \geq 3$ and $t_1 \geq 2$, we have $t_0t_1 \geq 4$. Because $a_{2i-1} \geq 0$, we conclude that $a_{2i+1} - a_{2i-1} \geq a_{2i-1} - a_{2i-3} \geq 1$. Similarly, since $a_{j+1} = t_{i+1}a_j - a_{j-1}$ for $j = 2i-1, 2i, 2i+1$ by (1.7), and since $a_{2i} \geq 0$ and $a_{2i} - a_{2i-2} \geq 2$, we get

$$a_{2i+2} - a_{2i} = (t_0t_1 - 4)a_{2i} + (a_{2i} - a_{2i-2}) \geq a_{2i} - a_{2i-2} \geq 2$$

by Lemma 3.2. Therefore, the statement holds for every $i \geq 1$. This proves (i).

By (i), we know that a_i is at least a_2 or a_3 for each $i \geq 2$. Since $a_2 = t_0 - 1 \geq 2$ and $a_3 = t_1(t_0 - 1) - 1 \geq 2$, it follows that $a_i \geq 2$ for each $i \geq 2$. This implies that $b_i \geq 1$ for each $i \geq 2$, for otherwise $a_i = t_{i+1}b_i + \xi_i \leq 1$. Therefore, we get (ii).

To prove (iii), it suffices to check that $a_i > t_i$ for $i = 3, 4$, since $a_{i+2j} \geq a_i$ by (i) and $t_i = t_{i+2j}$ for each $j \geq 0$. Since $t_0 \geq 4$ and $t_1 = 1$, or $t_0 \geq 3$ and $t_1 \geq 2$, we see that $a_3 = t_1(t_0 - 1) - 1$ is greater than $t_1 = t_3$. By (i), we have $a_4 \geq a_2 + 2 = t_0 + 1 > t_4$. Therefore, we get (iii).

By (iii), it follows that $t_i b_i + \xi_i = a_i \geq t_i + 1$ for each $i \geq 3$. Hence, we have $b_i \geq 1 + (1 - \xi_i)t_i^{-1}$. Since $\xi_i = 1$ if $i \equiv 0, 1 \pmod 4$, and $\xi_i = -1$ otherwise, we get $b_i \geq 1$ if $i \equiv 0, 1 \pmod 4$, and $b_i > 1$ otherwise. This proves (iv).

By (iii), we have $la_i > t_i$ for any $l \in \mathbf{N}$ if $i \geq 3$. Hence, (v) holds when $i \geq 3$. Suppose that $la_2 = t_2$ for some $l \in \mathbf{N}$. Then, we have $l(t_0 - 1) = t_0$. Hence, $t_0/(t_0 - 1) = l$ must be an integer. This contradicts that $t_0 \geq 3$. Therefore, we get (v). This completes the proof of Lemma 3.1.

4. Theory of local slice construction

The main part of Theorem 1.1 (i), and Theorem 1.5 (i) are proved by means of *local slice construction* due to Freudenburg [**7**]. In this section, we briefly review basic facts about locally nilpotent derivations, and summarize the theory of local slice construction.

Let $f, g \in k[\mathbf{x}]$ be such that $\ker D = k[f, g]$ for some $D \in \mathrm{LND}_k\, k[\mathbf{x}]$. Then, $\ker D$ has transcendence degree two over k (cf. [**20**, 1.4]). Hence, f and g are algebraically independent over k. Since $k[f, g]$ is the polynomial ring in f and g over k, it is clear that f is an irreducible element of $k[f, g]$, and $k[f]$ is factorially closed in $k[f, g]$, i.e., if pq belongs to $k[f]$ for $p, q \in k[f, g] \setminus \{0\}$, then p and q belong to $k[f]$. Recall that $\ker D$ is *factorially closed* in $k[\mathbf{x}]$ (cf. [**20**, 1.3.1]). Thus, it follows that f is an irreducible element of $k[\mathbf{x}]$, and $k[f]$ is factorially closed in $k[\mathbf{x}]$. In particular, $fk[\mathbf{x}]$ is a prime ideal of $k[\mathbf{x}]$. Since $g \not\approx f$, and g is also an irreducible element of $k[\mathbf{x}]$, we know that g does not belong to $fk[\mathbf{x}]$. Here, $p \approx q$ (resp. $p \not\approx q$) denotes that p and q are linearly dependent (resp. linearly independent) over k for $p, q \in k[\mathbf{x}]$.

We summarize these facts in the following lemma.

Lemma 4.1. *Let $f, g \in k[\mathbf{x}]$ be such that $\ker D = k[f, g]$ for some $D \in \mathrm{LND}_k\, k[\mathbf{x}]$. Then, f and g are algebraically independent over k, f is an irreducible element of $k[\mathbf{x}]$, $k[f]$ and $k[f, g]$ are factorially closed in $k[\mathbf{x}]$, and $fk[\mathbf{x}]$ is a prime ideal of $k[\mathbf{x}]$ to which g does not belong.*

In the situation of the lemma above, assume that K is an extension field of k. Then, D naturally extends to a locally nilpotent derivation $\bar{D} := \mathrm{id}_K \otimes D$ of $K[\mathbf{x}] := K \otimes_k k[\mathbf{x}]$. Since K is flat over k, we have $\ker \bar{D} = K \otimes_k \ker D = K[f, g]$. Hence, $f = 1 \otimes f$ is an irreducible element of $K[\mathbf{x}]$ (see also [**4**, Corollary 1.7] for a stronger statement). Similarly, $f + \alpha$ is an irreducible element of $K[\mathbf{x}]$ for each $\alpha \in K$, since $K[f + \alpha, g] = K[f, g]$.

Lemma 4.2. (i) *Assume that $f, g \in k[\mathbf{x}]$ are algebraically independent over k. If $\ker D = k[f, g]$ holds for some $D \in \mathrm{Der}_k\, k[\mathbf{x}]$, then we have $k[f] \cap gk[\mathbf{x}] = \{0\}$.*

(ii) *If* $f, s \in k[\mathbf{x}] \setminus k$ *are such that* $D(f) = 0$ *and* $D(s) \neq 0$ *for some* $D \in \mathrm{Der}_k\, k[\mathbf{x}]$*, then* f *and* s *are algebraically independent over* k*.*
(iii) *Let* $f, g, s \in k[\mathbf{x}]$ *be such that* $\ker D = k[f, g]$ *and* $D(s) \neq 0$ *for some* $D \in \mathrm{LND}_k\, k[\mathbf{x}]$*. Then,* $P := k[f, s] \cap gk[\mathbf{x}]$ *is a principal prime ideal of* $k[f, s]$*. If* $\eta(y, z)$ *is an irreducible element of* $k[y, z]$ *such that* $\eta(f, s)$ *belongs to* $gk[\mathbf{x}]$*, then* P *is generated by* $\eta(f, s)$*.*

Proof. (i) Suppose to the contrary that $k[f] \cap gk[\mathbf{x}] \neq \{0\}$. Then, we may find $\lambda(y) \in k[y] \setminus \{0\}$ and $g' \in k[\mathbf{x}] \setminus \{0\}$ such that $\lambda(f) = gg'$. Since $D(f) = D(g) = 0$, it follows that $gD(g') = D(\lambda(f)) = 0$. Hence, we get $D(g') = 0$. Thus, g' belongs to $\ker D$. Since $\ker D = k[f, g]$, we may write $g' = \mu(f, g)$, where $\mu(y, z) \in k[y, z] \setminus \{0\}$. Then, we have $\lambda(f) = gg' = g\mu(f, g)$. Since $\lambda(y)$ and $\mu(y, z)$ are nonzero, this contradicts that f and g are algebraically independent over k. Therefore, we get $k[f] \cap gk[\mathbf{x}] = \{0\}$.

(ii) Since k is of characteristic zero, $k(f, s)$ is a separable extension of $k(f)$. Since $D(f) = 0$ and $D(s) \neq 0$, it follows that $k(f, s)$ is not a finite extension of $k(f)$ (cf. [**18**, Proposition 5.2]). Hence, s is transcendental over $k(f)$. Since f does not belong to k by assumption, $k(f)$ is a transcendence extension of k. Therefore, f and s are algebraically independent over k.

(iii) Since D is locally nilpotent and $\ker D = k[f, g]$ by assumption, f and g are algebraically independent over k by Lemma 4.1. Hence, we get $k[f] \cap gk[\mathbf{x}] = \{0\}$ by (i). By Lemma 4.1, $gk[\mathbf{x}]$ is a prime ideal of $k[\mathbf{x}]$ to which f does not belong. Hence, $P = k[f, s] \cap gk[\mathbf{x}]$ is a prime ideal of $k[f, s]$. Let $\bar{f}$ be the image of f in the k-domain $k[\mathbf{x}]/gk[\mathbf{x}]$. Then, $\bar{f}$ is transcendental over k, since $k[f] \cap gk[\mathbf{x}] = \{0\}$. Hence, $k[f, s]/P$ has transcendence degree at least one over k. Accordingly, P is of height at most one. Since $k[f, s]$ is the polynomial ring in f and s over k by (ii), we see that $k[f, s]$ is a noetherian UFD. Therefore, P is a principal ideal of $k[f, s]$ (cf. [**19**, Theorem 20.1]).

Assume that $\eta(y, z)$ is an irreducible element of $k[y, z]$ for which $q := \eta(f, s)$ belongs to $gk[\mathbf{x}]$. Then, q belongs to P, and is an irreducible element of $k[f, s]$. Since P is a principal prime ideal of $k[f, s]$, it follows that P is generated by q. $\square$

Now, we briefly summarize the theory of local slice construction. Assume that $D \in \mathrm{LND}_k\, k[\mathbf{x}]$ is irreducible and satisfies the following conditions:

(LSC1) There exist $f, g \in k[\mathbf{x}]$ such that $D = \Delta_{(f,g)}$ and $\ker D = k[f, g]$;
(LSC2) There exist $s \in k[\mathbf{x}] \setminus gk[\mathbf{x}]$ and $F \in k[f] \setminus \{0\}$ such that $D(s) = gF$.

Since $\ker D = k[f, g]$ by (LSC1), we know that f and g are algebraically independent over k by Lemma 4.1. In particular, we have $g \neq 0$. Hence, we get $D(s) = gF \neq 0$ due to (LSC2). Thus, f and s are algebraically independent over k by Lemma 4.2 (ii), and $P := k[f, s] \cap gk[\mathbf{x}]$ is a principal prime ideal of $k[f, s]$ by Lemma 4.2 (iii).

We show that P is not the zero ideal. Since $D(g) = 0$, we see that D induces a derivation $\bar{D}$ of $R := k[\mathbf{x}]/gk[\mathbf{x}]$ over k. Then, $\bar{D}$ is nonzero, since $D(k[\mathbf{x}])$ is not contained in $gk[\mathbf{x}]$ by the irreducibility of D. Because $gk[\mathbf{x}]$ is a prime ideal of $k[\mathbf{x}]$ by Lemma 4.1, and is of height one, we know that R is a domain having transcendence degree two over k. Accordingly, $\ker \bar{D}$ has transcendence degree at most one over k. On the other hand, the images $\bar{f}$

and $\bar{s}$ of f and s in R belong to $\ker \bar{D}$, since $D(f) = 0$ and $D(s) = gF$. Thus, $\bar{f}$ and $\bar{s}$ are algebraically dependent over k. Consequently, some element of $k[f,s] \setminus \{0\}$ belongs to $gk[\mathbf{x}]$. Therefore, P is not the zero ideal.

Since P is a principal prime ideal of $k[f,s]$, we may find an irreducible element q of $k[f,s]$ such that $P = qk[f,s]$. Then, $h := g^{-1}q$ belongs to $k[\mathbf{x}]$. We note that q does not belong to $k[f]$. In fact, $q = gh$ belongs to $gk[\mathbf{x}] \setminus \{0\}$, and $k[f] \cap gk[\mathbf{x}] = \{0\}$ by Lemma 4.2 (i).

The following theorem is due to Freudenburg [**7**] (see also [**9**, Theorem 5.24]).

Theorem 4.3 (Freudenburg)**.** *In the notation above, we have the following assertions*:
(a) $\Delta_{(f,h)}$ *belongs to* $\mathrm{LND}_k\, k[\mathbf{x}]$.
(b) $\Delta_{(f,h)}(s) = -hF$.
(c) *If* $\Delta_{(f,h)}$ *is irreducible, then* $\ker \Delta_{(f,h)} = k[f,h]$.

The element $\Delta_{(f,h)}$ of $\mathrm{LND}_k\, k[\mathbf{x}]$ is called a locally nilpotent derivation obtained by *local slice construction* from the data (f,g,s).

5. Irreducibility criteria

In the situation of Theorem 4.3, the following proposition is useful to check the irreducibility of $\Delta_{(f,h)}$.

Proposition 5.1. *If* F *and* $\Delta_{(f,h)}(g_0)$ *have no common factor for some* $g_0 \in k[\mathbf{x}]$*, then* $\Delta_{(f,h)}$ *is irreducible.*

Proof. Suppose to the contrary that $E := \Delta_{(f,h)}$ is not irreducible. Then, there exists $p \in k[\mathbf{x}] \setminus k$ such that $E(k[\mathbf{x}])$ is contained in $pk[\mathbf{x}]$. Without loss of generality, we may assume that p is an irreducible element of $k[\mathbf{x}]$. Then, F does not belong to $pk[\mathbf{x}]$, since F and $E(g_0)$ have no common factor by assumption, and $E(g_0)$ belongs to $pk[\mathbf{x}]$ by supposition. By Theorem 4.3 (b), we have $-hF = E(s)$. Since $E(s)$ belongs to $pk[\mathbf{x}]$, it follows that h belongs to $pk[\mathbf{x}]$. Hence, $q = gh$ belongs to $pk[\mathbf{x}]$. Recall that q is an element of $k[f,s]$, and f and s are algebraically independent over k. Hence, we may consider the partial derivatives $\partial q/\partial f$ and $\partial q/\partial s$. Since $E(h) = E(f) = 0$ and $E(s) = -hF$, we get

$$E(g) = E(qh^{-1}) = E(q)h^{-1} = \left(\frac{\partial q}{\partial f}E(f) + \frac{\partial q}{\partial s}E(s)\right)h^{-1} = -\frac{\partial q}{\partial s}F$$

by chain rule. Because $E(g)$ belongs to $pk[\mathbf{x}]$, and F does not belong to $pk[\mathbf{x}]$ as mentioned, we conclude that $\partial q/\partial s$ belongs to $pk[\mathbf{x}]$.

Now, take $\phi(y,z) \in k[y,z]$ such that $\phi(f,s) = q$. Then, $\phi(y,z)$ is an irreducible element of $k[y,z]$ by the irreducibility of q in $k[f,s]$. Since q does not belong to $k[f]$, it follows that $\phi(y,z)$ does not belong to $k[y]$. Let $\bar{f}$ and $\bar{s}$ denote the images of f and s in $k[\mathbf{x}]/pk[\mathbf{x}]$, respectively. We show that $\bar{f}$ is algebraic over k. Define elements of the polynomial ring $(k[\mathbf{x}]/pk[\mathbf{x}])[z]$ by $\psi(z) := \phi(\bar{f}, z)$ and $\psi'(z) := d\psi(z)/dz$. Then, we have $\psi(\bar{s}) = \psi'(\bar{s}) = 0$, since $\phi(f,s) = q$ and $(\partial\phi/\partial z)(f,s) = \partial q/\partial s$ belong to $pk[\mathbf{x}]$ as mentioned. Now, suppose to the contrary that $\bar{f}$ is transcendental over k. Then, $\bar{f}$ and z are algebraically independent over k, since z is an indeterminate over

$k[\mathbf{x}]/pk[\mathbf{x}]$. Because $\phi(y,z)$ is an irreducible element of $k[y,z]$ not belonging to $k[y]$, it follows that $\psi(z)$ is an irreducible element of $k[\bar{f},z]$ not belonging to $k[\bar{f}]$. Consequently, $\psi(z)$ is an irreducible polynomial in z over $k[\bar{f}]$, and hence over $k(\bar{f})$. Since k is of characteristic zero, this contradicts that $\psi(\bar{s}) = \psi'(\bar{s}) = 0$. Therefore, $\bar{f}$ is algebraic over k.

Let $\mu_1(y)$ be the minimal polynomial of $\bar{f}$ over k. Then, we have $\mu_1(\bar{f}) = 0$. Hence, $\mu_1(f)$ belongs to $pk[\mathbf{x}]$. On the other hand, $\mu_1(f)$ is an irreducible element of $k[f]$, since f is not a constant. By Lemma 4.1, $k[f]$ is factorially closed in $k[\mathbf{x}]$. Hence, it follows that $\mu_1(f)$ is an irreducible element of $k[\mathbf{x}]$. Therefore, we conclude that $\mu_1(f) \approx p$.

By definition, $\psi(z)$ belongs to $k[\bar{f}][z]$. We claim that $\psi(z)$ is nonzero. In fact, if $\psi(z) = 0$, then $\phi(y,z)$ is divisible by $\mu_1(y)$. By the irreducibility of $\phi(y,z)$, it follows that $\phi(y,z) \approx \mu_1(y)$. Hence, $\phi(y,z)$ belongs to $k[y]$, a contradiction. Thus, $\psi(z)$ belongs to $k[\bar{f}][z] \setminus \{0\}$. Since $\psi(\bar{s}) = 0$ and $\bar{f}$ is algebraic over k, this implies that $\bar{s}$ is algebraic over k. Let $\mu_2(z)$ be the minimal polynomial of $\bar{s}$ over k. Then, we have $\mu_2(s) = ph_0$ for some $h_0 \in k[\mathbf{x}]$, while $\mu_2'(s)$ does not belong to $pk[\mathbf{x}]$, where $\mu_2'(z) := d\mu_2(z)/dz$. Since $p \approx \mu_1(f)$ is killed by D, and $D(s) = gF$ by (LSC2), we get

$$pD(h_0) = D(ph_0) = D(\mu_2(s)) = \mu_2'(s)D(s) = \mu_2'(s)gF.$$

Because $\mu_2'(s)$ and F do not belong to $pk[\mathbf{x}]$, this implies that g belongs to $pk[\mathbf{x}]$. By the irreducibility of g, it follows that $g \approx p$. Since $p \approx \mu_1(f)$, we conclude that f and g are algebraically independent over k, a contradiction. Therefore, E must be irreducible. □

To prove the irreducibility of polynomials in two variables, we use the following lemma.

Lemma 5.2. *Let $q \in k[x_1,x_2]$ be such that $q^{\mathbf{v}} = x_1^a + \alpha x_2^b$ for some $\mathbf{v} \in \mathbf{N}^2$, $a,b \in \mathbf{N}$ with $\gcd(a,b) = 1$ and $\alpha \in k^\times$. Then, q is an irreducible element of $k[x_1,x_2]$.*

PROOF. Since $\gcd(a,b) = 1$ and $\alpha \neq 0$ by assumption, we see that $x_1^a + \alpha x_2^b$ is an irreducible element of $k[x_1,x_2]$. Suppose to the contrary that $q = q_1q_2$ for some $q_1,q_2 \in k[x_1,x_2] \setminus k$. Then, $q_1^{\mathbf{v}}$ and $q_2^{\mathbf{v}}$ belong to $k[x_1,x_2] \setminus k$ by the choice of $\mathbf{v}$. Since $x_1^a + \alpha x_2^b = q^{\mathbf{v}} = q_1^{\mathbf{v}}q_2^{\mathbf{v}}$, this contradicts the irreducibility of $x_1^a + \alpha x_2^b$. Therefore, q is an irreducible element of $k[x_1,x_2]$. □

Let Γ be a totally ordered additive group. Then, we have the following lemma.

Lemma 5.3. *Let $\mathbf{v} = (a,t) \in \Gamma^2$ be such that $a > 0$ or $t > 0$. Then, for each $i \geq 0$, we have*

$$\eta_i(x_1,x_2)^{\mathbf{v}} = \begin{cases} x_1^{t_i} & \text{if} \quad t_ia > a_it \\ x_1^{t_i} + x_2^{a_i} & \text{if} \quad t_ia = a_it \\ x_2^{a_i} & \text{if} \quad t_ia < a_it. \end{cases}$$

PROOF. Assume that $t \leq 0$. Then, we have $a > 0$ by assumption. Hence, we get $t_ia > 0 \geq a_it$. By definition, $\eta_i(x_1,x_2)$ is a monic polynomial in x_1 over $k[x_2]$ of degree $t_i \geq 1$. Since $a > 0$ and $t \leq 0$, it follows that $\eta_i(x_1,x_2)^{\mathbf{v}} = x_1^{t_i}$. Therefore, the assertion is true.

Assume that $t > 0$. If $i \equiv 0, 1 \pmod 4$, then we have $a_i = t_i b_i + 1$. Hence, we know that

$$\begin{aligned}\deg_{\mathbf{v}} x_1^j x_2^{(t_i-j)b_i} &= ja + (t_i - j)b_i t = ja + (t_i - j)\left(\frac{a_i - 1}{t_i}\right)t \\ &= a_i t + \frac{t}{t_i}(j - t_i) + \frac{j}{t_i}(t_i a - a_i t)\end{aligned}$$

for $j = 1, \ldots, t_i$. If $t_i a \geq a_i t$, then $\deg_{\mathbf{v}} x_1^j x_2^{(t_i-j)b_i}$ has the maximum value $\deg_{\mathbf{v}} x_1^{t_i} = t_i a$ when $j = t_i$. Hence, we have

$$\left(\eta_i(x_1, x_2) - x_2^{t_i b_i + 1}\right)^{\mathbf{v}} = \left(\sum_{j=1}^{t_i} \alpha_j^i x_1^j x_2^{(t_i-j)b_i}\right)^{\mathbf{v}} = x_1^{t_i}.$$

Since $x_2^{t_i b_i + 1} = x_2^{a_i}$, it follows that $\eta_i(y, z)^{\mathbf{v}} = x_1^{t_i}$ if $t_i a > a_i t$, and $\eta_i(y, z)^{\mathbf{v}} = x_1^{t_i} + x_2^{a_i}$ if $t_i a = a_i t$. If $t_i a < a_i t$, then $\deg_{\mathbf{v}} x_1^j x_2^{(t_i-j)b_i}$ is less than $a_i t = \deg_{\mathbf{v}} x_2^{a_i}$ for $j = 1, \ldots, t_i$. Hence, we get $\eta_i(y, z)^{\mathbf{v}} = x_2^{a_i}$. Therefore, the lemma is true when $i \equiv 0, 1 \pmod 4$.

If $i \equiv 2, 3 \pmod 4$, then we have $a_i = t_i b_i - 1$. Hence, we know that

$$\begin{aligned}\deg_{\mathbf{v}} x_1^{t_i-j} x_2^{jb_i-1} &= (t_i - j)a + (jb_i - 1)t = (t_i - j)a + j\frac{a_i + 1}{t_i}t - t \\ &= t_i a + \frac{t}{t_i}(j - t_i) + \frac{j}{t_i}(a_i t - t_i a)\end{aligned}$$

for $j = 1, \ldots, t_i$. If $t_i a \leq a_i t$, then $\deg_{\mathbf{v}} x_1^{t_i-j} x_2^{jb_i-1}$ has the maximum value $\deg_{\mathbf{v}} x_2^{t_i b_i - 1} = a_i t$ when $j = t_i$. This implies that

$$\left(\eta_i(x_1, x_2) - x_1^{t_i}\right)^{\mathbf{v}} = \left(\sum_{j=1}^{t_i} \alpha_j^i x_1^{t_i-j} x_2^{jb_i-1}\right)^{\mathbf{v}} = x_2^{t_i b_i - 1} = x_2^{a_i}.$$

Hence, we have $\eta_i(y, z)^{\mathbf{v}} = x_2^{a_i}$ if $t_i a < a_i t$, and $\eta_i(y, z)^{\mathbf{v}} = x_1^{t_i} + x_2^{a_i}$ if $t_i a = a_i t$. If $t_i a > a_i t$, then $\deg_{\mathbf{v}} x_1^{t_i-j} x_2^{jb_i-1}$ is less than $t_i a = \deg_{\mathbf{v}} x_1^{t_i}$ for $j = 1, \ldots, t_i$. Hence, we get $\eta_i(y, z)^{\mathbf{v}} = x_1^{t_i}$. Therefore, the lemma is true when $i \equiv 2, 3 \pmod 4$. $\square$

By Lemma 5.3, we have $\eta_i(x_1, x_2)^{\mathbf{v}_i} = x_1^{t_i} + x_2^{a_i}$ for $\mathbf{v}_i := (a_i, t_i)$ for each $i \geq 0$. Moreover, t_i and $a_i = t_i b_i + \xi_i$ are mutually prime, since $\xi_i = 1, -1$. Therefore, we conclude that $\eta_i(x_1, x_2)$ is an irreducible element of $k[x_1, x_2]$ by Lemma 5.2.

For each $i \geq 3$, we define

$$\tilde{h}_i = \eta_i(x_1, x_2\lambda(x_1)^{-1})\lambda(x_1)^{a_i}.$$

Then, $\tilde{h}_i$ belongs to $k[x_1, x_2]$, since $\eta_i(x_1, x_2)$ is a monic polynomial in x_2 over $k[x_1]$ of degree a_i. We show that $\tilde{h}_i$ is an irreducible element of $k[x_1, x_2]$. Let $\tilde{\mathbf{v}}_i = (a_i, ua_i + t_i)$, where $u := \deg_y \lambda(y)$. Then, we have $\deg_{\tilde{\mathbf{v}}_i} x_1 = a_i = \deg_{\mathbf{v}_i} x_1$ and $\deg_{\tilde{\mathbf{v}}_i} x_2\lambda(x_1)^{-1} = t_i = \deg_{\mathbf{v}_i} x_2$. Since $\eta_i(x_1, x_2)^{\mathbf{v}_i} = x_1^{t_i} + x_2^{a_i}$, we see that

$$\eta_i\left(x_1, x_2\lambda(x_1)^{-1}\right)^{\tilde{\mathbf{v}}_i} = x_1^{t_i} + \left(\left(x_2\lambda(x_1)^{-1}\right)^{\tilde{\mathbf{v}}_i}\right)^{a_i} = x_1^{t_i} + \left(x_2(cx_1^u)^{-1}\right)^{a_i},$$

where c is the leading coefficient of $\lambda(y)$. Hence, we get $\tilde{h}_i^{\tilde{\mathbf{v}}_i} = c^{a_i} x_1^{ua_i+t_i} + x_2^{a_i}$. Since t_i and a_i are mutually prime, it follows that $ua_i + t_i$ and a_i are mutually prime. Therefore, $\tilde{h}_i$ is an irreducible element of $k[x_1, x_2]$ thanks to Lemma 5.2.

Since $\deg_z \theta(z) = t_0 - 1 = a_2$, we see that

$$\tilde{h}_2 := \tilde{\eta}_2 \left(x_1, x_2\lambda(x_1)^{-1}\right) \lambda(x_1)^{a_2} = \Big(x_1 + \theta\big(x_2\lambda(x_1)^{-1}\big)\Big)\lambda(x_1)^{a_2}$$

belongs to $k[x_1, x_2]$. We show that $\tilde{h}_2$ is an irreducible element of $k[x_1, x_2]$. Let $\tilde{\mathbf{v}}_2 = (a_2, ua_2 + 1)$. Then, we have $\deg_{\tilde{\mathbf{v}}_2} x_1 = a_2$ and $\deg_{\tilde{\mathbf{v}}_2} x_2\lambda(x_1)^{-1} = 1$. Hence, we get

$$\tilde{h}_2^{\tilde{\mathbf{v}}_2} = \left(x_1 + \left(x_2(cx_1^u)^{-1}\right)^{a_2}\right)(cx_1^u)^{a_2} = c^{a_2} x_1^{ua_2+1} + x_2^{a_2}.$$

Since ua_2+1 and a_2 are mutually prime, we conclude that $\tilde{h}_2$ is an irreducible element of $k[x_1, x_2]$ thanks to Lemma 5.2.

The following lemma is also a consequence of Lemma 5.3.

Lemma 5.4. *Let $f, p \in k(\mathbf{x}) \setminus \{0\}$ and $\mathbf{w} \in \Gamma^3$ be such that $\deg_{\mathbf{w}} f > 0$ or $\deg_{\mathbf{w}} p > 0$. If $t_i \deg_{\mathbf{w}} f \neq a_i \deg_{\mathbf{w}} p$ for $i \in \mathbf{Z}_{\geq 0}$, then we have*

$$\deg_{\mathbf{w}} \eta_i(f, p) = \max\{t_i \deg_{\mathbf{w}} f, a_i \deg_{\mathbf{w}} p\}.$$

PROOF. Set $\mathbf{v} = (\deg_{\mathbf{w}} f, \deg_{\mathbf{w}} p)$. First, assume that $t_i \deg_{\mathbf{w}} f > a_i \deg_{\mathbf{w}} p$. Then, we have $\eta_i(x_1, x_2)^{\mathbf{v}} = x_1^{t_i}$ by Lemma 5.3, since $\deg_{\mathbf{w}} f > 0$ or $\deg_{\mathbf{w}} p > 0$ by assumption. This implies that $\eta_i(f, p)^{\mathbf{w}} = (f^{\mathbf{w}})^{t_i}$. Hence, we get $\deg_{\mathbf{w}} \eta_i(f, p) = t_i \deg_{\mathbf{w}} f$. Thus, the lemma is true in the case where $t_i \deg_{\mathbf{w}} f > a_i \deg_{\mathbf{w}} p$. Next, assume that $t_i \deg_{\mathbf{w}} f \leq a_i \deg_{\mathbf{w}} p$. Then, we have $t_i \deg_{\mathbf{w}} f < a_i \deg_{\mathbf{w}} p$, since $t_i \deg_{\mathbf{w}} f \neq a_i \deg_{\mathbf{w}} p$ by assumption. By Lemma 5.3, it follows that $\eta_i(x_1, x_2)^{\mathbf{v}} = x_2^{a_i}$. This implies that $\eta_i(f, p)^{\mathbf{w}} = (p^{\mathbf{w}})^{a_i}$. Hence, we get $\deg_{\mathbf{w}} \eta_i(f, p) = a_i \deg_{\mathbf{w}} p$. Thus, the lemma is true in the case where $t_i \deg_{\mathbf{w}} f \leq a_i \deg_{\mathbf{w}} p$, and therefore in all cases. ☐

6. Local slice constructions (I)

The goal of this section is to prove Theorem 1.1 (i), except for (a) when $(t_0, t_1, i) = (3, 1, 4)$. The exceptional case is postponed to Section 11. At the end of this section, we also prove Proposition 1.4.

Consider the following statements for $i \in \{0\} \cup I$:

(1) f_i and f_{i+1} belong to $k[\mathbf{x}] \setminus \{0\}$, D_i belongs to $\mathrm{LND}_k\, k[\mathbf{x}]$, $D_i(r) = f_i f_{i+1}$, and r does not belong to $f_i k[\mathbf{x}]$. If $i \geq 1$, then $-D_i$ is obtained by a local slice construction from the data (f_i, f_{i-1}, r).
(2) If $i \neq \max I$, then D_i is irreducible and $\ker D_i = k[f_i, f_{i+1}]$.
(3) If $i \neq \max I$, then q_{i+1} is an irreducible element of $k[f_{i+1}, r]$ belonging to $f_i k[\mathbf{x}]$.
(4) If $i \geq 2$ and $i \neq \max I$, then f_i and $D_i(f_{i-2})$ have no common factor.

We note that, if $i \neq \max I$, then (1), (2) and (3) imply that

$$k[f_{i+1}, r] \cap f_i k[\mathbf{x}] = q_{i+1} k[f_{i+1}, r]. \tag{6.1}$$

To see this, it suffices to check that the assumptions of Lemma 4.2 (iii) are fulfilled for $D = D_i$, $f = f_{i+1}$, $g = f_i$, $s = r$ and $\eta(y, z) = \eta_{i+1}(y, z)$. By (2), we have $\ker D_i = k[f_i, f_{i+1}]$. By (1), we have $D_i(r) = f_i f_{i+1} \neq 0$,

and D_i belongs to $\mathrm{LND}_k\, k[\mathbf{x}]$. Moreover, $\eta_{i+1}(y,z)$ is an irreducible element of $k[y,z]$ as mentioned after Lemma 5.3, and $\eta_{i+1}(f_{i+1},r)=q_{i+1}$ belongs to $f_ik[\mathbf{x}]$ by (3). Thus, the assumptions of Lemma 4.2 (iii) are fulfilled. Therefore, we get (6.1). Similarly, if $i\neq \max I$ and f_{i-1} belongs to $k[\mathbf{x}]$, then (1) and (2) imply that

$$k[f_i,r]\cap f_{i+1}k[\mathbf{x}]=q_ik[f_i,r]. \tag{6.2}$$

Actually, since $q_i=\eta_i(f_i,r)=f_{i-1}f_{i+1}$ belongs to $f_{i+1}k[\mathbf{x}]$, we obtain (6.2) by applying Lemma 4.2 (iii) with $D=D_i$, $f=f_i$, $g=f_{i+1}$, $s=r$ and $\eta(y,z)=\eta_i(y,z)$.

We prove the following proposition using the theory of local slice construction.

Proposition 6.1. *The statements* (1) *through* (4) *hold for each* $i\in\{0\}\cup I$.

Proof. We proceed by induction on i. First, assume that $i=0$. Recall that $f_0=x_2$, $f_1=x_1$ and $D_0=\partial/\partial x_3$. Since r has the form $x_1x_2x_3+h$ for some $h\in k[x_1,x_2]\setminus x_2k[\mathbf{x}]$, we have $D_0(r)=x_1x_2$, and r does not belong to $x_2k[\mathbf{x}]$. From these conditions, we know that (1) and (2) are true. As for (3), we see from (1.1) that q_1 is an irreducible element of $k[x_1,r]$ belonging to $x_2k[\mathbf{x}]$. Since $i<2$, (4) is obvious. Therefore, (1) through (4) hold for $i=0$.

Next, take any $j\in I$, and assume that (1) through (4) hold for $i<j$. First, we prove that (1) holds for $i=j$. Put $l=j-1$ and $l'=j+1$. Then, (1) through (4) hold for $i=l$ by induction assumption. By (1), it follows that f_l and f_j belong to $k[\mathbf{x}]\setminus\{0\}$, D_l belongs to $\mathrm{LND}_k\, k[\mathbf{x}]$, $D_l(r)=f_lf_j$, and r does not belong to $f_lk[\mathbf{x}]$. Since j is an element of I, we have $l\neq\max I$. Hence, D_l is irreducible and $\ker D_l=k[f_l,f_j]$ by (2). Since $D_l=\Delta_{(f_j,f_l)}$ by definition, we know that D_l satisfies (LSC1) for $f=f_j$ and $g=f_l$, and (LSC2) for $s=r$ and $F=f_j$. By (3), q_j is an irreducible element of $k[f_j,r]$ belonging to $f_lk[\mathbf{x}]$. Hence, $f_{l'}=q_jf_l^{-1}$ belongs to $k[\mathbf{x}]\setminus\{0\}$. This proves the first part of (1). By (a) and (b) of Theorem 4.3, we know that $D_j=\Delta_{(f_{l'},f_j)}=-\Delta_{(f_j,f_{l'})}$ belongs to $\mathrm{LND}_k\, k[\mathbf{x}]$ and satisfies

$$D_j(r)=(-\Delta_{(f_j,f_{l'})})(r)=-(-f_{l'}f_j)=f_jf_{l'},$$

and $-D_j=\Delta_{(f_j,f_{l'})}$ is obtained by a local slice construction from the data (f_j,f_l,r). Finally, we prove that r does not belong to $f_jk[\mathbf{x}]$. Suppose to the contrary that $r=f_jg'$ for some $g'\in k[\mathbf{x}]$. Then, g' does not belong to k, for otherwise $f_jf_{l'}=D_j(r)=D_j(f_j)g'=0$. Since $\ker D_l=k[f_l,f_j]$, we see that f_j also does not belong to k by Lemma 4.1. Hence, r is not an irreducible element of $k[\mathbf{x}]$. On the other hand, r is a linear and primitive polynomial in x_3 over $k[x_1,x_2]$. Hence, r is an irreducible element of $k[\mathbf{x}]$, a contradiction. Thus, r does not belong to $f_jk[\mathbf{x}]$. Therefore, (1) holds for $i=j$.

Next, we prove that (2) holds for $i=j$. So assume that $j\neq\max I$. Then, we have $t_0\geq 2$, since $I=\{1\}$ if $t_0=1$. In view of Theorem 4.3 (c), it suffices to check that D_j is irreducible. Since $t_0\geq 2$, we know that D_1 is irreducible as mentioned after (1.4). Hence, the assertion is true if $j=1$. So assume that $j\geq 2$. We prove that (4) holds for $i=j$. Then, it follows that D_j is irreducible thanks to Proposition 5.1, since $F=f_j$. Because D_l

is an element of $\mathrm{LND}_k\, k[\mathbf{x}]$ with $\ker D_l = k[f_l, f_j]$, we know that $\mathfrak{p}_j := f_j k[\mathbf{x}]$ is a prime ideal of $k[\mathbf{x}]$ by Lemma 4.1. Therefore, it suffices to prove that $D_j(f_{j-2})$ does not belong to $\mathfrak{p}_j$.

Since (1) holds for $i \leq j$, we have $D_i(r) = f_i f_{i+1} \neq 0$ for each $i \leq j$. On the other hand, $D_i = \Delta_{(f_{i+1}, f_i)}$ kills f_i and f_{i+1} for any $i \geq 0$. Hence, we know that r and f_i are algebraically independent over k for $i \leq l'$ by Lemma 4.2 (ii). Thus, we may regard q_i as a polynomial in r and f_i over k for each $i \leq l'$. First, we show that

$$q := a_j \frac{\partial q_l}{\partial f_l} f_l + \frac{\partial q_l}{\partial r} r \tag{6.3}$$

does not belong to $\mathfrak{p}_j$. Suppose to the contrary that q belongs to $\mathfrak{p}_j$. Then, q belongs to $\mathfrak{p}' := \mathfrak{p}_j \cap k[f_l, r]$. Since $j \geq 2$, we have $l - 1 = j - 2 \geq 0$. Hence, (1) holds for $i = l - 1$, and so f_{l-1} belongs to $k[\mathbf{x}]$. Since (1) and (2) hold for $i = l$, it follows that $\mathfrak{p}' = q_l k[f_l, r]$ by (6.2) with $i = l$. Thus, $q' := q - a_l q_l$ belongs to $q_l k[f_l, r]$. Write $q_l = \eta_l(f_l, r) = f_l^{t_l} + r^{a_l} + h$, where $h \in k[f_l, r]$. Then, we can easily check that $\deg_{f_l} h < t_l$. When $l \equiv 0, 1 \pmod 4$, we have $\deg_r h \leq (t_l - 1)b_l < b_l t_l + 1 = a_l$. When $l \equiv 2, 3 \pmod 4$, we have $l \geq 2$, and so $b_l \geq 1$ by Lemma 3.1 (ii). Hence, we get

$$\deg_r h \leq (t_l - 1)b_l - 1 < t_l b_l - 1 = a_l.$$

Thus, we may write

$$\begin{aligned} a_j \frac{\partial q_l}{\partial f_l} f_l &= a_j \left(t_l f_l^{t_l - 1} + \frac{\partial h}{\partial f_l} \right) f_l = a_j t_l f_l^{t_l} + h_1 \\ \frac{\partial q_l}{\partial r} r &= \left(a_l r^{a_l - 1} + \frac{\partial h}{\partial r} \right) r = a_l r^{a_l} + h_2, \end{aligned}$$

where $h_1, h_2 \in k[f_l, r]$ are such that $\deg_{f_l} h_i < t_l$ and $\deg_r h_i < a_l$ for $i = 1, 2$. Since $t_l a_j - a_l = a_{l'}$ by (1.7), it follows that

$$\begin{aligned} q' = q - a_l q_l &= (a_j t_l f_l^{t_l} + h_1) + (a_l r^{a_l} + h_2) - a_l(f_l^{t_l} + r^{a_l} + h) \\ &= (t_l a_j - a_l) f_l^{t_l} + h_1 + h_2 - a_l h = a_{l'} f_l^{t_l} + h_1 + h_2 - a_l h. \end{aligned} \tag{6.4}$$

From this, we see that $\deg_r q' < a_l = \deg_r q_l$. Since q' belongs to $q_l k[f_l, r]$ as mentioned, it follows that $q' = 0$. On the other hand, we have $a_{l'} = a_{j+1} \neq 0$ by the assumption that $j \neq \max I$. Hence, we see from (6.4) that $\deg_{f_l} q' = t_l$, a contradiction. Therefore, we conclude that q does not belong to $\mathfrak{p}_j$.

By chain rule, we have

$$D_j(q_i) = \frac{\partial q_i}{\partial f_i} D_j(f_i) + \frac{\partial q_i}{\partial r} D_j(r)$$

for each i. Since $D_j = \Delta_{(f_{l'}, f_j)}$ kills $f_{l'}$ and f_j, and since $D_j(r) = f_j f_{l'}$ by (1) with $i = j$, it follows that

$$D_j(f_l) = D_j(q_j f_{l'}^{-1}) = D_j(q_j) f_{l'}^{-1} = \frac{\partial q_j}{\partial r} D_j(r) f_{l'}^{-1} = \frac{\partial q_j}{\partial r} f_j. \tag{6.5}$$

Similarly, we have

$$\begin{aligned} D_j(f_{j-2}) &= D_j(q_l f_j^{-1}) = D_j(q_l) f_j^{-1} \\ &= \left(\frac{\partial q_l}{\partial f_l} D_j(f_l) + \frac{\partial q_l}{\partial r} D_j(r) \right) f_j^{-1} = \frac{\partial q_l}{\partial f_l} \frac{\partial q_j}{\partial r} + \frac{\partial q_l}{\partial r} f_{l'}, \end{aligned} \tag{6.6}$$

where we use (6.5) for the last equality. Put $(y) = yk[y,z]$. Then, we have $\eta_j(y,z) \equiv z^{a_j} \pmod{(y)}$, and so $\partial\eta_j(y,z)/\partial z \equiv a_j z^{a_j-1} \pmod{(y)}$. Since $f_j k[f_j, r]$ is contained in $\mathfrak{p}_j$, it follows that

$$(6.7) \qquad f_l f_{l'} = q_j \equiv r^{a_j} \pmod{\mathfrak{p}_j} \quad \text{and} \quad \frac{\partial q_j}{\partial r} \equiv a_j r^{a_j-1} \pmod{\mathfrak{p}_j}.$$

Thus, we know by (6.6) that $f_l D_j(f_{j-2})$ is congruent to

$$\frac{\partial q_l}{\partial f_l} f_l(a_j r^{a_j-1}) + \frac{\partial q_l}{\partial r} r^{a_j} = q r^{a_j-1}$$

modulo $\mathfrak{p}_j$. Since r does not belong to $\mathfrak{p}_j$ by (1) with $i = j$, and q does not belong to $\mathfrak{p}_j$ as shown above, we conclude that $D_j(f_{j-2})$ does not belong to $\mathfrak{p}_j$. This proves that (4) holds for $i = j$, and thereby proving that (2) holds for $i = j$.

Finally, we prove that (3) holds for $i = j$. Since $f_{l'}$ and r are algebraically independent over k as mentioned, we know that $q_{l'} = \eta_{l'}(f_{l'}, r)$ is an irreducible element of $k[f_{l'}, r]$ by the irreducibility of $\eta_{l'}(y,z)$ in $k[y,z]$. We show that $q_{l'}$ belongs to $\mathfrak{p}_j$. By (1) with $i = j$, we see that r does not belong to $\mathfrak{p}_j$. Since $\ker D_l = k[f_l, f_j]$ by (2) with $i = l$, we know that f_l does not belong to $\mathfrak{p}_j$ in view of Lemma 4.1. Since $j \neq \max I$ by the assumption of (3), we have $\ker D_j = k[f_j, f_{l'}]$ by (2) with $i = j$. Hence, $f_{l'}$ also does not belong to $\mathfrak{p}_j$ by Lemma 4.1. Since $b_{l'} = t_j b_j - b_l + \xi_j$ by definition, we have

$$a_j = t_j b_j + \xi_j = b_l + b_{l'}.$$

Hence, we get $f_l f_{l'} \equiv r^{b_l + b_{l'}} \pmod{\mathfrak{p}_j}$ by the first part of (6.7). This gives that

$$(6.8) \qquad f_l r^{-b_l} \equiv f_{l'}^{-1} r^{b_{l'}}, \quad f_l^{-1} r^{b_l} \equiv f_{l'} r^{-b_{l'}} \pmod{\mathfrak{p}_j k[\mathbf{x}]_{\mathfrak{p}_j}}.$$

Put $\theta_i(z) = \sum_{m=1}^{t_i} \alpha_m^i z^m$ for each $i \geq 0$. Then, we have

$$(6.9) \qquad \begin{aligned} \eta_i(f_i, r) &= r^{t_i b_i}(r + \theta_i(f_i r^{-b_i})) &&\text{if } i \equiv 0,1 \pmod 4 \\ \eta_i(f_i, r) &= f_i^{t_i} r^{-1}(r + \theta_i(f_i^{-1} r^{b_i})) &&\text{if } i \equiv 2,3 \pmod 4, \end{aligned}$$

and $r = x_1x_2x_3 - \theta_0(x_2) - \theta_1(x_1)$. We show that $\eta_l(f_l, r)$ belongs to $\mathfrak{p}_j$. First, assume that $j = 1$. Then, we have $l = 0$. Since $b_0 = 0$, we see from (6.9) that

$$\eta_0(f_0, r) = r + \theta_0(x_2) = x_1x_2x_3 - \theta_1(x_1).$$

Hence, $\eta_0(f_0, r)$ belongs to $\mathfrak{p}_1 = x_1 k[\mathbf{x}]$. Next, assume that $j \geq 2$. Then, f_{j-2} belongs to $k[\mathbf{x}]$ by (1) with $i = j-2$. Hence, $\eta_l(f_l, r) = f_{j-2} f_j$ belongs to $\mathfrak{p}_j$. Thus, $\eta_l(f_l, r)$ belongs to $\mathfrak{p}_j$ in all cases. Thanks to (6.9), it follows that $r + \theta_l(f_l r^{-b_l})$ and $r + \theta_l(f_l^{-1} r^{b_l})$ belong to $\mathfrak{p}_j k[\mathbf{x}]_{\mathfrak{p}_j}$ if $l \equiv 0,1 \pmod 4$ and if $l \equiv 2,3 \pmod 4$, respectively. Assume that $l \equiv 0,1 \pmod 4$. Then, this implies that $r + \theta_l(f_{l'}^{-1} r^{b_{l'}})$ belongs to $\mathfrak{p}_j k[\mathbf{x}]_{\mathfrak{p}_j}$ by (6.8). Since $l' = l + 2$, we have $\theta_{l'}(z) = \theta_l(z)$ and $l' \equiv 2,3 \pmod 4$. Hence, we see from the second equality of (6.9) that $q_{l'} = \eta_{l'}(f_{l'}, r)$ belongs to $\mathfrak{p}_j k[\mathbf{x}]_{\mathfrak{p}_j}$. Since $\mathfrak{p}_j k[\mathbf{x}]_{\mathfrak{p}_j} \cap k[\mathbf{x}] = \mathfrak{p}_j$, it follows that $q_{l'}$ belongs to $\mathfrak{p}_j$. Similarly, we can check that $q_{l'}$ belongs to $\mathfrak{p}_j$ when $l \equiv 2,3 \pmod 4$. Therefore, (3) holds for $i = j$. This proves that (1) through (4) hold for every $i \in \{0\} \cup I$. $\square$

As a consequence of Proposition 6.1, we know that (6.1) holds for each $i \in \{0\} \cup I$ with $i \neq \max I$, and (6.2) holds for each $i \in I$ with $i \neq \max I$. This proves the first part of the following lemma.

Lemma 6.2. *Assume that* $(j, l) = (i, i-1)$ *for some* $i \in I$, *or* $(j, l) = (i-1, i)$ *for some* $i \in I \setminus \{1\}$. *Then, we have*

$$k[f_j, r] \cap f_l k[\mathbf{x}] = q_j k[f_j, r].$$

If $a_j \geq 2$, *then we have* $(r + k[f_j]) \cap f_l k[\mathbf{x}] = \emptyset$.

Here, for $s \in k[\mathbf{x}]$ and a k-vector subspace A of $k[\mathbf{x}]$, we define

$$s + A = \{s + f \mid f \in A\}.$$

The last part of the lemma is proved as follows. Suppose to the contrary that there exists $h \in (r + k[f_j]) \cap f_l k[\mathbf{x}]$. Then, we have $\deg_r h = 1$, and so $h \neq 0$. Since h belongs to $k[f_j, r] \cap f_l k[\mathbf{x}] = q_j k[f_j, r]$, it follows that $\deg_r h \geq \deg_r q_j = a_j \geq 2$, a contradiction. Therefore, we get $(r + k[f_j]) \cap f_l k[\mathbf{x}] = \emptyset$.

Now, let us complete the proof of Theorem 1.1 (i), except for (a) when $(t_0, t_1, i) \neq (3, 1, 4)$. Proposition 6.1, we know that (1) through (4) hold for each $i \in \{0\} \cup I$. By (1), we get the first part of Theorem 1.1 (i). By (2), we get (b) of Theorem 1.1 (i). Hence, it remains only to check (a) of Theorem 1.1 (i) in the cases where $t_0 = i = 1$ and $(t_0, t_1, i) = (2, 1, 2)$. If $t_0 = 1$, then D_1 is not irreducible and $\ker D_1 \neq k[f_1, f_2]$ as mentioned after (1.4). Hence, (a) holds when $t_0 = i = 1$. Assume that $(t_0, t_1) = (2, 1)$. Then, we have $D_2 = \tau_2 \circ D_1' \circ \tau_2^{-1}$ by Theorem 1.1 (ii). Since $t_1 = 1$, we know that D_1' is not irreducible and $\ker D_1' \neq k[f_1', f_2']$. Hence, it follows that D_2 is not irreducible and $\ker D_2 \neq k[f_2, f_3]$. Thus, (a) holds when $(t_0, t_1, i) = (2, 1, 2)$. This proves Theorem 1.1 (i), except for (a) when $(t_0, t_1, i) \neq (3, 1, 4)$.

Finally, we prove Proposition 1.4. So assume that $\alpha_j^i = 0$ for $i = 0, 1$ and $j = 1, \ldots, t_i - 1$. Then, we have $q_i = \eta_i(f_i, r) = f_i^{t_i} + r^{a_i}$ for each $i \geq 1$. Set $d_i = \deg_{\mathbf{t}} f_i$ for each i. We prove that f_{i+1} is $\mathbf{t}$-homogeneous and $t_{i+1} d_{i+1} = t_0 t_1 a_{i+1}$ for each $i \in \{0\} \cup I$ by induction on i. Since $f_1 = x_1$, $d_1 = t_0$ and $a_1 = 1$, we see that the statement holds for $i = 0$. Since f_2 is $\mathbf{t}$-homogeneous as mentioned, and $d_2 = t_1(t_0 - 1)$ and $a_2 = t_0 - 1$, we see that the statement also holds for $i = 1$. So assume that $i \geq 2$. Then, f_l is $\mathbf{t}$-homogeneous and $t_l d_l = t_0 t_1 a_l$ for $l = i-1, i$ by induction assumption. Hence, we have $\deg_{\mathbf{t}} f_i^{t_i} = t_i d_i = t_0 t_1 a_i = \deg_{\mathbf{t}} r^{a_i}$. Since f_i and r are $\mathbf{t}$-homogeneous, this implies that $q_i = f_i^{t_i} + r^{a_i}$ is $\mathbf{t}$-homogeneous. Because f_{i-1} is $\mathbf{t}$-homogeneous, it follows that $f_{i+1} = q_i f_{i-1}^{-1}$ is $\mathbf{t}$-homogeneous. Note that $\deg_{\mathbf{t}} q_i = \deg_{\mathbf{t}} f_i^{t_i} = t_i d_i$. Since $q_i = f_{i-1} f_{i+1}$, we get $d_{i-1} + d_{i+1} = t_i d_i$. Hence, we know that

$$\begin{aligned} t_{i+1} d_{i+1} - t_0 t_1 a_{i+1} &= t_{i+1}(t_i d_i - d_{i-1}) - t_0 t_1 (t_{i+1} a_i - a_{i-1}) \\ &= t_{i+1}(t_i d_i - t_0 t_1 a_i) - (t_{i-1} d_{i-1} - t_0 t_1 a_{i-1}) \end{aligned}$$

in view of (1.7). Since $t_l d_l = t_0 t_1 a_l$ for $l = i-1, i$, the right-hand side of the preceding equality is zero. Thus, we get $t_{i+1} d_{i+1} = t_0 t_1 a_{i+1}$. This proves that the statement holds for every $i \in \{0\} \cup I$. Therefore, f_i and f_{i+1} are $\mathbf{t}$-homogeneous for each $i \in \{0\} \cup I$. This completes the proof of Proposition 1.4.

7. Local slice constructions (II)

In this section, we prove Theorem 1.5 (i). So assume that $i = 2$ and $t_0 \geq 3$, or $i \geq 3$, $t_0 \geq 3$ and $(t_0, t_1) \neq (3, 1)$. Then, $l := i-1$ belongs to I, and is not the maximum of I. By Theorem 1.1 (i), it follows that $D_l = \Delta_{(f_i, f_l)}$ is irreducible and locally nilpotent, and satisfies $\ker D_l = k[f_l, f_i]$. Hence, D_l satisfies (LSC1) for $f = f_i$ and $g = f_l$.

We show that D_l satisfies (LSC2) for $s = r_i$, and for $F = \lambda(f_2)$ if $i = 2$, and $F = \lambda(f_i)f_i$ if $i \geq 3$. First, we check that r_i does not belong to $\mathfrak{p}_l := f_l k[\mathbf{x}]$. Suppose to the contrary that $r_i = \lambda(f_i)\tilde{r} - \mu(f_i, f_l)$ belongs to $\mathfrak{p}_l$. Then, $\lambda(f_i)\tilde{r}$ belongs to $\mathfrak{p}_l$, since $\mu(f_i, f_l)$ belongs to $f_l k[f_i, f_l]$ by the choice of $\mu(y, z)$. Since $\ker D_l = k[f_l, f_i]$, we know by Lemmas 4.1 and 4.2 (i) that $\mathfrak{p}_l$ is a prime ideal of $k[\mathbf{x}]$ with $k[f_i] \cap \mathfrak{p}_l = \{0\}$. Because f_i is not a constant, we have $\lambda(f_i) \neq 0$ by the assumption that $\lambda(y) \neq 0$. Hence, $\lambda(f_i)$ does not belong to $\mathfrak{p}_l$. Thus, $\tilde{r}$ belongs to $\mathfrak{p}_l$. However, $\tilde{r} = x_2$ does not belong to $\mathfrak{p}_1 = x_1 k[\mathbf{x}]$ if $i = 2$, and $\tilde{r} = r$ does not belong to $\mathfrak{p}_l$ if $i \geq 3$ by (1) of Proposition 6.1. This is a contradiction. Therefore, r_i does not belong to $\mathfrak{p}_l$. If $i = 2$, then we have $D_l(\tilde{r}) = D_1(x_2) = x_1 = f_1$ by (1.4). If $i \geq 3$, then we have $D_l(\tilde{r}) = D_l(r) = f_l f_i$ by Theorem 1.1 (i). Since

$$D_l(r_i) = D_l\bigl(\lambda(f_i)\tilde{r} - \mu(f_i, f_l)\bigr) = \lambda(f_i) D_l(\tilde{r}),$$

it follows that $D_1(r_2) = f_1 \lambda(f_2)$ if $i = 2$, and $D_l(r_i) = f_l \lambda(f_i) f_i$ if $i \geq 3$. Thus, we get $D_l(r_i) = f_l F$ for each $i \geq 2$ for the F mentioned above. Therefore, D_l satisfies (LSC2) for $s = r_i$ and this F.

Recall that $\tilde{h}_i$ is an irreducible element of $k[x_1, x_2]$ as shown after Lemma 5.2. Since $D_l(f_i) = 0$ and $D_l(r_i) = f_l F \neq 0$, we know that f_i and r_i are algebraically independent over k by Lemma 4.2 (ii). Hence, it follows that

$$\tilde{q}_i := \tilde{h}_i(f_i, r_i) = \tilde{\eta}_i\left(f_i, r_i \lambda(f_i)^{-1}\right) \lambda(f_i)^{a_i} \tag{7.1}$$

is an irreducible element of $k[f_i, r_i]$. We show that $\tilde{q}_i$ belongs to $\mathfrak{p}_l$. Since $\mu(f_i, f_l)$ belongs to $\mathfrak{p}_l$, we have $r_i \equiv \lambda(f_i)\tilde{r} \pmod{\mathfrak{p}_l}$. Hence, we get

$$\tilde{q}_i \equiv \tilde{\eta}_i(f_i, \tilde{r}) \lambda(f_i)^{a_i} \pmod{\mathfrak{p}_l}.$$

If $i = 2$, then $\tilde{\eta}_2(f_2, x_2) = x_1 x_3$ belongs to $\mathfrak{p}_1 = x_1 k[\mathbf{x}]$. If $i \geq 3$, then $\tilde{\eta}_i(f_i, \tilde{r}) = \eta_i(f_i, r) = f_l f_{i+1}$ belongs to $\mathfrak{p}_l$. Therefore, it follows that $\tilde{q}_i$ belongs to $\mathfrak{p}_l$. Hence, $\tilde{f}_{i+1} = \tilde{q}_i f_l^{-1}$ belongs to $k[\mathbf{x}]$. From (a) and (b) of Theorem 4.3, we conclude that $\tilde{D}_i = \Delta_{(\tilde{f}_{i+1}, f_i)}$ belongs to $\mathrm{LND}_k\, k[\mathbf{x}]$, and $\tilde{D}_i(r_i) = F \tilde{f}_{i+1}$ is as in Theorem 1.5 (i).

In view of Theorem 4.3 (c), it remains only to prove that $\tilde{D}_i$ is irreducible to complete the proof of Theorem 1.5 (i).

Under the assumption of Theorem 1.5, the following lemma holds.

Lemma 7.1. *$\lambda(f_i)$ and $\tilde{D}_i(f_{i-2})$ have no common factor.*

PROOF. Suppose to the contrary that $\lambda(f_i)$ and $\tilde{D}_i(f_{i-2})$ have a common factor $p \in k[\mathbf{x}] \setminus k$. By Lemma 4.1, $k[f_i]$ is factorially closed in $k[\mathbf{x}]$, since $\ker D_l = k[f_l, f_i]$ by Theorem 1.1 (i). Hence, p belongs to $k[f_i]$. Thus, p is divisible by $f_i - \alpha$ for some $\alpha \in \bar{k}$, where $\bar{k}$ is an algebraic closure of k. Since p is a factor of $\lambda(f_i)$, we have $\lambda(\alpha) = 0$. By the assumption that $\lambda(y)$ and $\mu(y, z)$ have no common factor, it follows that $\mu(\alpha, z) \neq 0$. Since $\mu(y, z)$ is an

element of $zk[y,z]$, we may write $\mu(\alpha,z)=-z\nu(z)$, where $\nu(z)\in k[z]\setminus\{0\}$. Then, r_i is congruent to $\lambda(\alpha)\tilde{r}-\mu(\alpha,f_l)=f_l\nu(f_l)$ modulo $\mathfrak{p}:=(f_i-\alpha)\bar{k}[\mathbf{x}]$. Note that $\tilde{\eta}_i(y,z)$ is a monic polynomial in z over $k[y]$ of degree a_i for any $i\geq 2$. Hence, we see from (7.1) that $\tilde{q}_i$ is congruent to $r_i^{a_i}$, and hence to $f_l^{a_i}\nu(f_l)^{a_i}$ modulo $\mathfrak{p}$. Put $\psi(z)=z^{a_i-1}\nu(z)^{a_i}$. Then, it follows that

$$\tilde{f}_{i+1}=\tilde{q}_i f_l^{-1}\equiv\psi(f_l)\pmod{\mathfrak{p}}.$$

If $i=2$, then we have $a_2=t_0-1\geq 2$, since $t_0\geq 3$. If $i\geq 3$, then we have $a_i\geq 2$ by Lemma 3.1 (ii), since $t_0\geq 3$ and $(t_0,t_1)\neq(3,1)$. Hence, $\psi(z)$ is not a constant, and so the derivative $\psi'(z)$ is nonzero.

Now, regard $\Delta:=\Delta_{(f_i,f_{i-2})}$ as a derivation of $\bar{k}[\mathbf{x}]$. Then, we have $\Delta(f_i-\alpha)=0$. Hence, $\Delta(\mathfrak{p})$ is contained in $\mathfrak{p}$. Since $\tilde{f}_{i+1}\equiv\psi(f_l)\pmod{\mathfrak{p}}$, it follows that

$$\Delta(\tilde{f}_{i+1})\equiv\Delta(\psi(f_l))\pmod{\mathfrak{p}}.$$

Since $\Delta(\tilde{f}_{i+1})=\Delta_{(f_i,f_{i-2})}(\tilde{f}_{i+1})=\tilde{D}_i(f_{i-2})$, and $\tilde{D}_i(f_{i-2})$ is divisible by p by supposition, this implies that $\Delta(\psi(f_l))$ belongs to $\mathfrak{p}$. By chain rule, we have $\Delta(\psi(f_l))=\psi'(f_l)\Delta(f_l)$, in which

$$\Delta(f_l)=\Delta_{(f_i,f_{i-2})}(f_l)=-D_l(f_{i-2})=-\frac{\partial q_l}{\partial r}f_l$$

by (6.5). Hence, $\psi'(f_l)f_l(\partial q_l/\partial r)$ belongs to $\mathfrak{p}$. We show that $\partial q_l/\partial r$ belongs to $\mathfrak{p}$. As discussed after Lemma 4.1, we may extend D_l to a locally nilpotent derivation $\bar{D}_l$ of $\bar{k}[\mathbf{x}]$ such that

$$\ker\bar{D}_l=\bar{k}[f_l,f_i]=\bar{k}[f_l,f_i-\alpha]. \tag{7.2}$$

Hence, we know by Lemmas 4.1 and 4.2 (i) that $\mathfrak{p}$ is a prime ideal of $\bar{k}[\mathbf{x}]$ such that $\bar{k}[f_l]\cap\mathfrak{p}=\{0\}$. Since $\psi'(f_l)\neq 0$ as mentioned, it follows that $\psi'(f_l)f_l$ does not belong to $\mathfrak{p}$. Thus, we conclude that $\partial q_l/\partial r$ belongs to $\mathfrak{p}$. Note that $\bar{D}_l(\mathfrak{p})$ is contained in $\mathfrak{p}$, since $\bar{D}_l(f_i-\alpha)=0$. Hence, $D_l^j(\partial q_l/\partial r)$ belongs to $\mathfrak{p}$ for any $j\geq 0$. Since $q_l=\eta_l(f_l,r)$ is a monic polynomial in r over $k[f_l]$ of degree a_l, we have $\partial^{a_l}q_l/\partial r^{a_l}=a_l!$. Because $D_l(f_l)=D_l(f_i)=0$ and $D_l(r)=f_lf_i$, it follows that

$$\begin{aligned}D_l^{a_l-1}\left(\frac{\partial q_l}{\partial r}\right)&=D_l^{a_l-2}\left(\frac{\partial^2 q_l}{\partial r^2}D_l(r)\right)=D_l^{a_l-2}\left(\frac{\partial^2 q_l}{\partial r^2}f_lf_i\right)\\&=D_l^{a_l-2}\left(\frac{\partial^2 q_l}{\partial r^2}\right)f_lf_i=\cdots=\frac{\partial^{a_l}q_l}{\partial r^{a_l}}(f_lf_i)^{a_l-1}=a_l!(f_lf_i)^{a_l-1}\end{aligned}$$

by chain rule. Therefore, $a_l!(f_lf_i)^{a_l-1}$ belongs to $\mathfrak{p}$.

When $i=2$, we have $a_l!(f_lf_i)^{a_l-1}=1$, since $a_l=a_1=1$. This implies that $\mathfrak{p}=\bar{k}[\mathbf{x}]$, a contradiction. Assume that $i\geq 3$. Then, we have $a_l=a_{i-1}\geq 2$ by Lemma 3.1 (ii). Hence, f_l or f_i belongs to $\mathfrak{p}$. In view of (7.2), we know by Lemma 4.1 that f_l does not belong to $\mathfrak{p}$. Hence, f_i belongs to $\mathfrak{p}$. Thus, we get $\alpha=0$. Since f_i is an element of $k[\mathbf{x}]$, it follows that $\mathfrak{p}\cap k[\mathbf{x}]=f_i\bar{k}[\mathbf{x}]\cap k[\mathbf{x}]=f_ik[\mathbf{x}]$, to which $\partial q_l/\partial r$ belongs. Hence, $\partial q_l/\partial r$ belongs to $f_ik[\mathbf{x}]\cap k[f_l,r]$. By Lemma 6.2, we have $f_ik[\mathbf{x}]\cap k[f_l,r]=q_lk[f_l,r]$. Thus, $\partial q_l/\partial r$ is divisible by q_l. This implies that $\partial q_l/\partial r=0$. Hence, we get $a_l=\deg_r q_l=0$, a contradiction. Therefore, $\lambda(f_i)$ and $\tilde{D}_i(f_{i-2})$ have no common factor. $\square$

If $i = 2$, then $\lambda(f_2)$ and $\tilde{D}_2(f_0)$ have no common factor by Lemma 7.1. Since $F = \lambda(f_2)$, it follows that $\tilde{D}_2$ is irreducible by Proposition 5.1. Assume that $i \geq 3$. Then, we have $F = \lambda(f_i)f_i$. By Lemma 7.1, we know that $\lambda(f_i)$ and $\tilde{D}_i(f_{i-2})$ have no common factor. Hence, it suffices to prove that f_i and $\tilde{D}_i(f_{i-2})$ have no common factor by virtue of Proposition 5.1. Since f_i is an irreducible element of $k[\mathbf{x}]$, we verify that $\tilde{D}_i(f_{i-2})$ does not belong to $\mathfrak{p}_i = f_i k[\mathbf{x}]$.

If $\lambda(0) = 0$, then f_i is a factor of $\lambda(f_i)$. Since $\lambda(f_i)$ and $\tilde{D}_i(f_{i-2})$ have no common factor, it follows that $\tilde{D}_i(f_{i-2})$ does not belong to $\mathfrak{p}_i$. So assume that $\lambda(0) \neq 0$. If $\mu(0, z) = 0$, then $\mu(f_i, f_l)$ belongs to $\mathfrak{p}_i$. Since $\tilde{r} = r$, it follows that $r_i \equiv \lambda(f_i)r \pmod{\mathfrak{p}_i}$. Hence, we see from (7.1) that

$$\tilde{q}_i \equiv \eta_i(f_i, r)\lambda(f_i)^{a_i} \equiv \lambda(0)^{a_i} q_i \pmod{\mathfrak{p}_i}.$$

Since $\tilde{q}_i = f_l \tilde{f}_{i+1}$ and $q_i = f_l f_{i+1}$, it follows that $f_l(\tilde{f}_{i+1} - \lambda(0)^{a_i} f_{i+1})$ belongs to $\mathfrak{p}_i$. Because f_l does not belong to $\mathfrak{p}_i$ by Lemma 4.1, we conclude that $\tilde{f}_{i+1} - \lambda(0)^{a_i} f_{i+1}$ belongs to $\mathfrak{p}_i$. Write $\tilde{f}_{i+1} - \lambda(0)^{a_i} f_{i+1} = f_i g$, where $g \in k[\mathbf{x}]$. Then, we have

$$\tilde{D}_i = \Delta_{(\tilde{f}_{i+1}, f_i)} = \lambda(0)^{a_i} \Delta_{(f_{i+1}, f_i)} + f_i \Delta_{(g, f_i)} = \lambda(0)^{a_i} D_i + f_i \Delta_{(g, f_i)}.$$

Since $f_i \Delta_{(g,f_i)}(f_{i-2})$ belongs to $\mathfrak{p}_i$, we get $\tilde{D}_i(f_{i-2}) \equiv \lambda(0)^{a_i} D_i(f_{i-2}) \pmod{\mathfrak{p}_i}$. By (4) of Proposition 6.1, f_i and $D_i(f_{i-2})$ have no common factor. Since $\lambda(0) \neq 0$ by assumption, it follows that $\lambda(0)^{a_i} D_i(f_{i-2})$ does not belong to $\mathfrak{p}_i$. Therefore, $\tilde{D}_i(f_{i-2})$ does not belong to $\mathfrak{p}_i$.

Finally, assume that $\lambda(0) \neq 0$ and $\mu(0, z) \neq 0$. Then, we have $\mu_j(0) \neq 0$ for some $j \geq 1$. Put

$$g' = a_i \mu_z(0, f_l) f_l - \mu(0, f_l) = \sum_{j \geq 1} \mu_j(0)(j a_i - 1) f_l^j,$$

where

$$\mu_z(y, z) := \frac{\partial \mu(y, z)}{\partial z} = \sum_{j \geq 1} j \mu_j(y) z^{j-1}.$$

Then, we have $g' \neq 0$, since $a_i \geq 2$ by Lemma 3.1 (ii).

Now, we prove that $\tilde{D}_i(f_{i-2})$ does not belong to $\mathfrak{p}_i$ by contradiction. Suppose to the contrary that $\tilde{D}_i(f_{i-2})$ belongs to $\mathfrak{p}_i$. Then, we have the following claim. Here, we define $q \in k[f_l, r]$ as in (6.3) with $j = i$, and $g = g' \partial q_l / \partial r$.

Claim 7.2. *$\lambda(0)q + g$ belongs to $\mathfrak{p}_i$.*

First, we assume this claim, and derive a contradiction. Since $\lambda(0)q + g$ is an element of $k[f_l, r]$, it follows from Claim 7.2 that $\lambda(0)q + g$ belongs to $\mathfrak{p}_i \cap k[f_l, r]$. By Lemma 6.2, we have $\mathfrak{p}_i \cap k[f_l, r] = q_l k[f_l, r]$. Hence,

$$g_1 := \lambda(0)(q - a_l q_l) + g = (\lambda(0)q + g) - \lambda(0) a_l q_l$$

belongs to $q_l k[f_l, r]$. This implies that $\deg_r g_1 \geq \deg_r q_l$ or $g_1 = 0$. From (6.4), we see that $\deg_r(q - a_l q_l) < a_l = \deg_r q_l$. Since g' is an element of $k[f_l] \setminus \{0\}$, we have

$$\deg_r g = \deg_r g' \frac{\partial q_l}{\partial r} = \deg_r \frac{\partial q_l}{\partial r} < \deg_r q_l.$$

Hence, we get $\deg_r g_1 < \deg_r q_l$. Thus, we conclude that $g_1 = 0$. Therefore, we obtain $q - a_l q_l \approx g$.

First, assume that $i = 3$. Then, we have $b_l = b_2 = 1$, and so

$$q_l = q_2 = \eta_2(f_2, r) = f_2^{t_0} + \sum_{j=1}^{t_0} \alpha_j^0 f_2^{t_0-j} r^{j-1}.$$

Regard q_l as a polynomial in r over $k[f_2]$. Then, the coefficient of r^{t_0-2} in q_l is equal to f_2 multiplied by a constant. From (6.3), we see that the same holds for the coefficient of r^{t_0-2} in q, and hence for the coefficient of r^{t_0-2} in

$$q - a_l q_l \approx g = g' \frac{\partial q_l}{\partial r} = g' \sum_{j=2}^{t_0} \alpha_j^0 (j-1) f_2^{t_0-j} r^{j-2}.$$

Because $\alpha_{t_0}^0 (t_0 - 1) g' \neq 0$, it follows that $g' \approx f_2$. Hence, we get $\deg_{f_2}(q - a_l q_l) = t_0 - 1$. On the other hand, we have $\deg_{f_2}(q - a_l q_l) = \deg_{f_l}(q - a_l q_l) = t_l = t_0$ by (6.4). This is a contradiction.

Next, assume that $i \geq 4$. Regard $q_l = \eta_l(f_l, r)$ as a polynomial in r over $k[f_l]$. We show that r^{a_l-1} does not appear in q_l. When $l \equiv 0, 1 \pmod 4$, it is easy to see that $r^{a_l-1} = r^{t_l b_l}$ does not appear in q_l. When $l \equiv 2, 3 \pmod 4$, we have $b_l \geq 2$ by Lemma 3.1 (iv), since $l = i - 1 \geq 3$. From this, we know that $r^{a_l-1} = r^{t_l b_l - 2}$ does not appear in q_l. By (6.3), it follows that r^{a_l-1} does not appear in q, and hence in $q - a_l q_l \approx g$. Since $\deg_r q_l = a_l$, however, we see that r^{a_l-1} appears in $\partial q_l / \partial r$ with coefficient $a_l \neq 0$, and hence in g with coefficient $a_l g' \neq 0$. This is a contradiction. Therefore, we conclude that $\tilde{D}_i(f_{i-2})$ does not belong to $\mathfrak{p}_i$.

Finally, we prove Claim 7.2. Recall that f_i and r_i are algebraically independent over k. Hence, we may consider the partial derivatives of $\tilde{q}_i = \tilde{h}_i(f_i, r_i)$ in f_i and r_i. Since $f_l = \tilde{q}_i \tilde{f}_{i+1}^{-1}$, $\tilde{D}_i(f_i) = \tilde{D}_i(\tilde{f}_{i+1}) = 0$ and $\tilde{D}_i(r_i) = \lambda(f_i) f_i \tilde{f}_{i+1}$, we have

$$\tilde{D}_i(f_l) = \tilde{D}_i(\tilde{q}_i \tilde{f}_{i+1}^{-1}) = \tilde{D}_i(\tilde{q}_i)\tilde{f}_{i+1}^{-1} = \frac{\partial \tilde{q}_i}{\partial r_i} \tilde{D}_i(r_i) \tilde{f}_{i+1}^{-1} = \frac{\partial \tilde{q}_i}{\partial r_i} \lambda(f_i) f_i \tag{7.3}$$

by chain rule. Since $r_i = \lambda(f_i) r - \mu(f_i, f_l)$, it follows that

$$\begin{aligned}
\lambda(f_i) f_i \tilde{f}_{i+1} &= \tilde{D}_i(r_i) = \tilde{D}_i\big(\lambda(f_i) r - \mu(f_i, f_l)\big) \\
&= \lambda(f_i)\tilde{D}_i(r) - \mu_z(f_i, f_l)\tilde{D}_i(f_l) = \lambda(f_i)\tilde{D}_i(r) - \mu_z(f_i, f_l) \frac{\partial \tilde{q}_i}{\partial r_i} \lambda(f_i) f_i.
\end{aligned}$$

Since $\lambda(f_i) \neq 0$, this gives that

$$\tilde{D}_i(r) = f_i \left(\mu_z(f_i, f_l) \frac{\partial \tilde{q}_i}{\partial r_i} + \tilde{f}_{i+1} \right). \tag{7.4}$$

From (7.3) and (7.4), we obtain

$$\begin{aligned}
\tilde{D}_i(q_l) &= \frac{\partial q_l}{\partial f_l} \tilde{D}_i(f_l) + \frac{\partial q_l}{\partial r} \tilde{D}_i(r) \\
&= f_i \left(\frac{\partial q_l}{\partial f_l} \frac{\partial \tilde{q}_i}{\partial r_i} \lambda(f_i) + \frac{\partial q_l}{\partial r} \left(\mu_z(f_i, f_l) \frac{\partial \tilde{q}_i}{\partial r_i} + \tilde{f}_{i+1} \right) \right).
\end{aligned}$$

Since $\tilde{f}_{i+1} = \tilde{q}_i f_l^{-1}$, the right-hand side of this equality is written as $f_i f_l^{-1} h_1$, where

$$h_1 := f_l \frac{\partial \tilde{q}_i}{\partial r_i} \left(\lambda(f_i) \frac{\partial q_l}{\partial f_l} + \mu_z(f_i, f_l) \frac{\partial q_l}{\partial r} \right) + \tilde{q}_i \frac{\partial q_l}{\partial r}.$$

Because $\tilde{D}_i(f_{i-2})$ belongs to $\mathfrak{p}_i$ by assumption,

$$h_1 = f_i^{-1} f_l \tilde{D}_i(q_l) = f_i^{-1} f_l \tilde{D}_i(f_{i-2} f_i) = f_l \tilde{D}_i(f_{i-2})$$

belongs to $\mathfrak{p}_i$. Since $\eta_i(0, z) = z^{a_i}$, we see from (7.1) that

$$\tilde{q}_i \equiv \eta_i(0, \lambda(f_i)^{-1} r_i) \lambda(f_i)^{a_i} = r_i^{a_i} \pmod{f_i k[f_i, r_i]}. \tag{7.5}$$

This gives that

$$\frac{\partial \tilde{q}_i}{\partial r_i} \equiv a_i r_i^{a_i - 1} \pmod{f_i k[f_i, r_i]}.$$

Hence, we have $h_1 \equiv r_i^{a_i-1} h_2 \pmod{\mathfrak{p}_i}$, where

$$h_2 := a_i f_l \left(\lambda(0) \frac{\partial q_l}{\partial f_l} + \mu_z(0, f_l) \frac{\partial q_l}{\partial r} \right) + r_i \frac{\partial q_l}{\partial r}.$$

Since $a_l = a_{i-1} \geq 2$ by Lemma 3.1 (ii), we have $(r + k[f_l]) \cap \mathfrak{p}_i = \emptyset$ by Lemma 6.2. Since $\lambda(0) \neq 0$ by assumption, and $r_i \equiv \lambda(0) r - \mu(0, f_l) \pmod{\mathfrak{p}_i}$, it follows that r_i does not belong to $\mathfrak{p}_i$. Thus, we know that h_2 belongs to $\mathfrak{p}_i$. Now, observe that

$$\begin{aligned} \lambda(0) q + g &= \lambda(0) \left(a_i f_l \frac{\partial q_l}{\partial f_l} + r \frac{\partial q_l}{\partial r} \right) + \big(a_i \mu_z(0, f_l) f_l - \mu(0, f_l) \big) \frac{\partial q_l}{\partial r} \\ &= a_i f_l \left(\lambda(0) \frac{\partial q_l}{\partial f_l} + \mu_z(0, f_l) \frac{\partial q_l}{\partial r} \right) + (\lambda(0) r - \mu(0, f_l)) \frac{\partial q_l}{\partial r}. \end{aligned}$$

Since the right-hand side of this equality is congruent to h_2 modulo $\mathfrak{p}_i$, we conclude that $\lambda(0) q + g$ belongs to $\mathfrak{p}_i$. This proves Claim 7.2, and thereby completing the proof of Theorem 1.5 (i).

8. Recurrence equations

In what follows, let Γ be the totally ordered additive group $\mathbf{Z}^3$ equipped with the lexicographic order such that $\mathbf{e}_1 < \mathbf{e}_2 < \mathbf{e}_3$. From Γ^3, we take the weight $\mathbf{w} := (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, and define

$$\delta_i = \deg_{\mathbf{w}} f_i$$

for each $i \geq 0$. Then, we have

$$\delta_0 = (0, 1, 0), \quad \delta_1 = (1, 0, 0), \quad \delta_2 = (1, 0, 1), \quad \deg_{\mathbf{w}} r = (1, 1, 1). \tag{8.1}$$

We remark that, if

$$\gamma_i := t_i \delta_i - a_i \deg_{\mathbf{w}} r > 0$$

for $i \in \mathbf{N}$, then the $\mathbf{w}$-degree of $q_i = \eta_i(f_i, r)$ is equal to $\max\{t_i \delta_i, a_i \deg_{\mathbf{w}} r\} = t_i \delta_i$ by Lemma 5.4, since $\deg_{\mathbf{w}} r > 0$. When this is the case, we have

$$\delta_{i+1} = \deg_{\mathbf{w}} f_{i+1} = \deg_{\mathbf{w}} q_i f_{i-1}^{-1} = t_i \delta_i - \delta_{i-1}.$$

Proposition 8.1. *If $t_0 \geq 3$, then $\delta_{i+1} = t_i \delta_i - \delta_{i-1}$ holds for each $i \in I \setminus \{1\}$.*

PROOF. We prove that the following statements hold for each $i \in I$ by induction on i:

(1) If $(t_0, t_1) = (3, 1)$ and $i \neq \max I$, then we have $\gamma_{i+1} > 0$.
(2) If $(t_0, t_1) \neq (3, 1)$, then we have $\gamma_{i+1} > 0$. If furthermore $i \geq 2$, then we have $\gamma_{i+1} - \gamma_{i-1} > 0$.
(3) If $i \geq 2$, then we have $\delta_{i+1} = t_i\delta_i - \delta_{i-1}$ and $\gamma_{i+1} = t_{i+1}\gamma_i - \gamma_{i-1}$.

Since

$$\gamma_2 = t_2\delta_2 - a_2 \deg_{\mathbf{w}} r = t_0(1, 0, 1) - (t_0 - 1)(1, 1, 1) = (1, 1 - t_0, 1) > 0,$$

we see that the statements hold when $i = 1$. Take $i \in I$ with $i \geq 2$. Then, $i - 1$ belongs to I, and is not the maximum of I. Hence, we have $\gamma_i > 0$ by the induction assumption of (1) and (2). As remarked above, this implies that $\delta_{i+1} = t_i\delta_i - \delta_{i-1}$, proving the first part of (3). Since $a_{i+1} = t_{i+1}a_i - a_{i-1}$ by (1.7), and $t_{i+1} = t_{i-1}$, it follows that

$$\begin{aligned}\gamma_{i+1} &= t_{i+1}\delta_{i+1} - a_{i+1} \deg_{\mathbf{w}} r = t_{i+1}(t_i\delta_i - \delta_{i-1}) - (t_{i+1}a_i - a_{i-1}) \deg_{\mathbf{w}} r \\ &= t_{i+1}(t_i\delta_i - a_i \deg_{\mathbf{w}} r) - (t_{i-1}\delta_{i-1} - a_{i-1} \deg_{\mathbf{w}} r) = t_{i+1}\gamma_i - \gamma_{i-1}.\end{aligned}$$

This proves the second part of (3). Hence, we have $\gamma_{j+1} = t_{j+1}\gamma_j - \gamma_{j-1}$ for $j = 2, \ldots, i$, where the case $j < i$ is due to the induction assumption of (3). When $i = 2$, we have $\gamma_3 = t_1\gamma_2 - \gamma_1$, and so $\gamma_3 - \gamma_1 = t_1\gamma_2 - 2\gamma_1$. Since $\gamma_2 > 0$ and

$$\gamma_1 = t_1\delta_1 - a_1 \deg_{\mathbf{w}} r = t_1(1, 0, 0) - (1, 1, 1) = (t_1 - 1, -1, -1) < 0,$$

it follows that γ_3 and $\gamma_3 - \gamma_1$ are positive. Hence, (1) and (2) are true if $i = 2$. When $i = 3$, we have $\gamma_{j+1} = t_{j+1}\gamma_j - \gamma_{j-1}$ for $j = 2, 3$. This gives that

$$\gamma_4 - \gamma_2 = t_0\gamma_3 - 2\gamma_2 = t_0(t_1\gamma_2 - \gamma_1) - 2\gamma_2 = (t_0t_1 - 2)\gamma_2 - t_0\gamma_1.$$

Since $t_0 \geq 3$ by assumption, and $\gamma_2 > 0$ and $\gamma_1 < 0$, it follows that $\gamma_4 - \gamma_2 > 0$, and so $\gamma_4 > 0$. Thus, (1) and (2) are true if $i = 3$. When $(t_0, t_1) = (3, 1)$, we have $I = \{1, \ldots, 4\}$. Hence, the proof is completed in this case. So assume that $(t_0, t_1) \neq (3, 1)$ and $i \geq 4$. Then, we have $\gamma_{j+1} = t_{j+1}\gamma_j - \gamma_{j-1}$ for $j = i - 2, i - 1, i$, since $i - 2 \geq 2$. Hence, we get

$$\gamma_{i+1} - \gamma_{i-1} = (t_0t_1 - 4)\gamma_{i-1} + (\gamma_{i-1} - \gamma_{i-3})$$

by Lemma 3.2. Since $(t_0, t_1) \neq (3, 1)$, and $t_0 \geq 3$ by assumption, we have $t_0t_1 \geq 4$. Since $i - 2 \geq 2$, we know that γ_{i-1} and $\gamma_{i-1} - \gamma_{i-3}$ are positive by the induction assumption of (2). Thus, we conclude that $\gamma_{i+1} - \gamma_{i-1} > 0$, and so $\gamma_{i+1} > 0$. Therefore, (2) is true if $i \geq 4$. This proves that (1), (2) and (3) hold for every $i \in I$. Consequently, we know by (3) that $\delta_{i+1} = t_i\delta_i - \delta_{i-1}$ holds for each $i \in I \setminus \{1\}$. □

We derive some consequences of Proposition 8.1. When $(t_0, t_1) = (3, 1)$, we have

$$\delta_3 = 3\delta_2 - \delta_1 = (2, 0, 3), \quad \delta_4 = \delta_3 - \delta_2 = (1, 0, 2), \quad \delta_5 = 3\delta_4 - \delta_3 = (1, 0, 3)$$

by Proposition 8.1 and (8.1). From this and (8.1), we can easily check that the following proposition holds when $(t_0, t_1) = (3, 1)$, since $I = \{1, \ldots, 4\}$.

Proposition 8.2. *If $t_0 \geq 3$, then the following assertions hold*:
(i) $\delta_{i+1} > 0$ *for each* $i \in \{0\} \cup I$.
(ii) δ_i *and* δ_{i+1} *are linearly independent for each* $i \in I$.
(iii) *If* $t_1 \geq 2$, *then* $\delta_{i+1} - \delta_i > 0$ *for each* $i \geq 1$.
(iv) *If* $(t_0, t_1) \neq (3, 1)$, *then* $\delta_{i+2} - \delta_i > 0$ *for each* $i \geq 1$.
(v) *The second component of* δ_{i+1} *is zero for each* $i \in \{0\} \cup I$.

Proof. By the discussion above, we may assume that $(t_0, t_1) \neq (3, 1)$. Then, we have $I = \mathbf{N}$, since $t_0 \geq 3$ by assumption.

First, we prove (v) by induction on i. By (8.1), the second components of δ_0 and δ_1 are zero. Assume that $i \geq 2$. Then, we have $\delta_{i+1} = t_i\delta_i - \delta_{i-1}$ by Proposition 8.1. Hence, the second component of δ_{i+1} is zero, proving (v).

We prove (i) through (iv) simultaneously by induction on i. When $i = 0$, (ii), (iii) and (iv) are obvious. Since $\delta_1 > 0$ by (8.1), we get (i). When $i = 1$, (i), (ii) and (iii) follow from (8.1). By Proposition 8.1, we have $\delta_3 - \delta_1 = t_0\delta_2 - 2\delta_1$. Since the third component of $t_0\delta_2 - 2\delta_1$ is $t_0 > 0$, we know that $\delta_3 - \delta_1 > 0$. This proves (iv). Assume that $i \geq 2$. Then, we have $\delta_{i-1} > 0$ and $\delta_{i+1} - \delta_{i-1} > 0$ by the induction assumption of (i) and (iv), since $i - 1 \geq 1$. Hence, we get $\delta_{i+1} > 0$, proving (i). By the induction assumption of (ii), δ_{i-1} and δ_i are linearly independent. Hence, δ_i and $\delta_{i+1} = t_i\delta_i - \delta_{i-1}$ are linearly independent, proving (ii). To show (iii), assume that $t_1 \geq 2$. Then, we have $t_i - 2 \geq 0$ independently of the parity of i. By the induction assumption of (i) and (iii), we have $\delta_i > 0$ and $\delta_i - \delta_{i-1} > 0$. Since $\delta_{i+1} = t_i\delta_i - \delta_{i-1}$, it follows that

$$\delta_{i+1} - \delta_i = (t_i - 2)\delta_i + (\delta_i - \delta_{i-1}) \geq \delta_i - \delta_{i-1} > 0,$$

proving (iii). Finally, we show (iv). When $i = 2$, we have

$$\delta_4 - \delta_2 = t_1\delta_3 - 2\delta_2 = t_1(t_0\delta_2 - \delta_1) - 2\delta_2 = (t_0t_1 - 2)\delta_2 - t_1\delta_1. \tag{8.2}$$

Since the third component of $(t_0t_1 - 2)\delta_2 - t_1\delta_1$ is $t_0t_1 - 2 > 0$, we know that $\delta_4 - \delta_2 > 0$. Thus, (iv) is true if $i = 2$. Assume that $i \geq 3$. Then, we have $\delta_{j+1} = t_j\delta_j - \delta_{j-1}$ for $j = i-1, i, i+1$, since $i - 1 \geq 2$. Hence, we get

$$\delta_{i+2} - \delta_i = (t_0t_1 - 4)\delta_i + (\delta_i - \delta_{i-2})$$

by Lemma 3.2. Since $t_0 \geq 3$ and $(t_0, t_1) \neq (3, 1)$ by assumption, we have $t_0t_1 \geq 4$. By the induction assumption of (i) and (iv), we have $\delta_i > 0$ and $\delta_i - \delta_{i-2} > 0$, since $i \geq 3$. Thus, it follows that $\delta_{i+2} - \delta_i > 0$, proving (iv). Therefore, (i) through (iv) hold for every i. □

9. Wildness (I)

In this section, we give a sufficient condition for wildness of certain exponential automorphisms. Assume that $i \geq 2$, and let $g, s \in k[\mathbf{x}]$ and $\nu(y) \in k[y] \setminus \{0\}$ be such that $E := \Delta_{(g,f_i)}$ is locally nilpotent, and

$$f_{i-1}g = q := \tilde{\eta}_i(f_i, s')\nu(f_i)^{a_i}, \quad \text{where} \quad s' := s\nu(f_i)^{-1}.$$

Take any $h \in k[f_i, g] \setminus \{0\}$, and set

$$\phi = \exp hE, \quad \alpha = \deg_{\mathbf{w}} \phi(s) \quad \text{and} \quad \beta = \deg_{\mathbf{w}} \phi(\tilde{r}).$$

Then, we have $\phi(f_i) = f_i$, since $E(f_i) = 0$. For each $j \geq 0$, we define

$$\epsilon_j = \deg_{\mathbf{w}} \phi(f_j) \quad \text{and} \quad d(j) = t_j \epsilon_j - a_j \beta.$$

Then, we have $\epsilon_i = \delta_i$, since $\phi(f_i) = f_i$. Because $\phi(\tilde{r})$ and $\phi(f_j)$ are not constants, and all the components of $\mathbf{w}$ are positive, we have $\beta > 0$, and $\epsilon_j > 0$ for each $j \geq 0$.

In this situation, consider the following conditions:

(a) δ_i and $\delta := \deg_{\mathbf{w}} g$ are linearly independent.
(b) Set $v = \deg_y \nu(y)$. Then, we have $\alpha \geq v\delta_2 + \delta$ and $\delta > \delta_2$ if $i = 2$, and $\alpha = (v + v_1)\delta_i + v_2\delta$ for some $v_1, v_2 \in \mathbf{N}$ if $i \geq 3$.
(c) If $i = 2$, then β and ϵ_1 are linearly independent. If $i \geq 3$, then we have $\beta \geq \alpha$.
(d) If $i \geq 3$, then we have $d(i-1) \neq 0$.
(e) If $i \geq 3$ and $d(i-1) < 0$, then we have $\beta \geq \epsilon_{i-1} - v\delta_i$ and $\delta > a_i v \delta_i$, and $a_{i-1}\beta - \delta_i$ and ϵ_{i-1} are linearly independent.

We mention that (a) implies that f_i and g are algebraically independent over k, and hence implies that E is nonzero.

In the notation above, we have the following theorem.

Theorem 9.1. *Assume that $i = 2$ and $t_0 \geq 3$, or $i \geq 3$, $t_0 \geq 3$ and $(t_0, t_1) \neq (3, 1)$. Let $g, s \in k[\mathbf{x}]$, $\nu(y) \in k[y] \setminus \{0\}$ and $h \in k[f_i, g] \setminus \{0\}$ be such that* (a) *through* (e) *are fulfilled. Then, $\phi = \exp hE$ is wild.*

First, we show that (b) implies

$$\epsilon_{i-1} = a_i \alpha - \delta \tag{9.1}$$

when $i = 2$ and $t_0 \geq 3$, or $i \geq 3$, $t_0 \geq 3$ and $(t_0, t_1) \neq (3, 1)$. Since $E(g) = 0$, we have $\phi(q) = \phi(f_{i-1}g) = \phi(f_{i-1})g$. Hence, we get $\deg_{\mathbf{w}} \phi(q) = \epsilon_{i-1} + \delta$. Thus, it suffices to prove that $\deg_{\mathbf{w}} \phi(q) = a_i\alpha$.

Since $\phi(f_i) = f_i$ and $\deg_y \nu(y) = v$, we have

$$\alpha' := \deg_{\mathbf{w}} \phi(s') = \deg_{\mathbf{w}} \phi(s\nu(f_i)^{-1}) = \deg_{\mathbf{w}} \phi(s)\nu(f_i)^{-1} = \alpha - v\delta_i.$$

First, assume that $i = 2$ and $t_0 \geq 3$. Then, (b) implies that

$$\alpha' \geq (v\delta_2 + \delta) - v\delta_2 = \delta > \delta_2. \tag{9.2}$$

Hence, $\deg_{\mathbf{w}} \theta(\phi(s')) = (t_0 - 1)\alpha'$ is greater than $\deg_{\mathbf{w}} f_2 = \delta_2$. Since $\phi(f_2) = f_2$ and $t_0 - 1 = a_2$, it follows that

$$\begin{aligned} \deg_{\mathbf{w}} \phi(q) &= \deg_{\mathbf{w}} \tilde{\eta}_2(f_2, \phi(s'))\nu(f_2)^{a_2} = \deg_{\mathbf{w}} \big(f_2 + \theta(\phi(s'))\big)\nu(f_2)^{a_2} \\ &= (t_0 - 1)\alpha' + a_2 v \delta_2 = a_2(\alpha' + v\delta_2) = a_2\alpha. \end{aligned}$$

This proves (9.1).

Next, assume that $i \geq 3$, $t_0 \geq 3$ and $(t_0, t_1) \neq (3, 1)$. Then, we have $a_i > t_i$ by Lemma 3.1 (iii), and (b) implies that

$$\alpha' = \big((v + v_1)\delta_i + v_2\delta\big) - v\delta_i = v_1\delta_i + v_2\delta \geq \delta_i + \delta.$$

Hence, it follows that

$$a_i\alpha' - t_i\delta_i \geq a_i(\delta_i + \delta) - t_i\delta_i = (a_i - t_i)\delta_i + a_i\delta > 0. \tag{9.3}$$

Thus, we get $\deg_{\mathbf{w}} \eta_i(f_i, \phi(s')) = a_i\alpha'$ by applying Lemma 5.4 with $f = f_i$ and $p = \phi(s')$. Since $\tilde{\eta}_i(y,z) = \eta_i(y,z)$, we know that

$$\deg_{\mathbf{w}} \phi(q) = \deg_{\mathbf{w}} \eta_i(f_i, \phi(s'))\nu(f_i)^{a_i} = a_i\alpha' + a_i v\delta_i = a_i\alpha,$$

proving (9.1). As a consequence, it follows that

$$\epsilon_{i-1} = a_i\alpha - \delta = a_i(v+v_1)\delta_i + (a_iv_2 - 1)\delta, \tag{9.4}$$

since $\alpha = (v+v_1)\delta_i + v_2\delta$ by (b).

Now, let us prove Theorem 9.1. Recall the notion of W-test polynomial introduced in Section 1 (see Definition 1.1). We show that f_2 is a W-test polynomial if $t_0 \geq 3$ by means of Proposition 1.4. Take any totally ordered additive group Λ, and $\mathbf{u} \in (\Lambda_{>0})^3$. Then, we have $\deg_{\mathbf{u}} x_2^i < \deg_{\mathbf{u}} x_2^{t_0-1}$ for $i = 0, \ldots, t_0 - 2$. Hence, $f_2^{\mathbf{u}}$ must be x_1x_3 or $-x_2^{t_0-1}$ or $x_1x_3 - x_2^{t_0-1}$. Since $t_0 \geq 3$, we see that these three polynomials are not divisible by $x_i - g$ for any $i \in \{1,2,3\}$ and $g \in k[\mathbf{x} \setminus \{x_i\}] \setminus k$, and by $x_i^{s_i} - cx_j^{s_j}$ for any $i,j \in \{1,2,3\}$ with $i \neq j$, $s_i, s_j \in \mathbf{N}$ and $c \in k^\times$. Therefore, f_2 is a W-test polynomial by Proposition 1.4.

First, assume that $i = 2$ and $t_0 \geq 3$. Then, we have

$$\deg_{\mathbf{w}} \phi(x_1) = \epsilon_1 = a_2\alpha - \delta = (t_0-1)\alpha - \delta \geq 2\alpha' - \delta > \delta_2 = \epsilon_2 = \deg_{\mathbf{w}} \phi(f_2)$$

by (9.1) and (9.2). Since $\deg_{\mathbf{w}} \phi(x_2) = \deg_{\mathbf{w}} \phi(\tilde{r}) = \beta$, we know by (c) that $\deg_{\mathbf{w}} \phi(x_1)$ and $\deg_{\mathbf{w}} \phi(x_2)$ are linearly independent. Thus, we conclude that ϕ is wild because f_2 is a W-test polynomial. This proves Theorem 9.1 when $i = 2$ and $t_0 \geq 3$.

Next, assume that $i \geq 3$, $t_0 \geq 3$ and $(t_0, t_1) \neq (3,1)$. Then, by Proposition 9.2 to follow, we know that $\deg_{\mathbf{w}} \phi(f_2) = \epsilon_2$ is less than $\deg_{\mathbf{w}} \phi(x_2) = \epsilon_0$, and $\deg_{\mathbf{w}} \phi(x_2) = \epsilon_0$ and $\deg_{\mathbf{w}} \phi(x_1) = \epsilon_1$ are linearly independent. Hence, we conclude that ϕ is wild similarly. Thus, the proof of Theorem 9.1 is completed.

Proposition 9.2. *In the situation of Theorem 9.1, assume that $t_0 \geq 3$, $(t_0, t_1) \neq (3,1)$ and $i \geq 3$. Set $i' = i$ if $d(i-1) > 0$, and $i' = i - 1$ if $d(i-1) < 0$. Then, the following statements hold for each $l \in \{1, \ldots, i'\}$:*
(i) *We have $d(l-1) > 0$, and ϵ_{l-1} and ϵ_l are linearly independent.*
(ii) *If $l \neq i'$, then we have $\epsilon_{l-1} = t_l\epsilon_l - \epsilon_{l+1}$ and $\epsilon_{l-1} > \epsilon_{l+1}$.*

Proof. First, we show that $d(l) > 0$ implies $\epsilon_{l-1} = t_l\epsilon_l - \epsilon_{l+1}$ for $l \geq 1$. Since $i \geq 3$, we have $\tilde{r} = r$. Hence, we know that $t_l \deg_{\mathbf{w}} \phi(f_l) > a_l \deg_{\mathbf{w}} \phi(r)$ by the assumption that $d(l) = t_l \deg_{\mathbf{w}} \phi(f_l) - a_l \deg_{\mathbf{w}} \phi(\tilde{r})$ is positive. By applying Lemma 5.4 with $f = \phi(f_l)$ and $p = \phi(r)$, we obtain

$$\deg_{\mathbf{w}} \phi\bigl(\eta_l(f_l, r)\bigr) = \deg_{\mathbf{w}} \eta_l\bigl(\phi(f_l), \phi(r)\bigr) = t_l \deg_{\mathbf{w}} \phi(f_l) = t_l\epsilon_l.$$

Since $\eta_l(f_l, r) = f_{l-1}f_{l+1}$, it follows that

$$\epsilon_{l-1} + \epsilon_{l+1} = \deg_{\mathbf{w}} \phi(f_{l-1}f_{l+1}) = \deg_{\mathbf{w}} \phi\bigl(\eta_l(f_l, r)\bigr) = t_l\epsilon_l.$$

Therefore, we get $\epsilon_{l-1} = t_l\epsilon_l - \epsilon_{l+1}$.

Similarly, if $d(i-1) < 0$, then we have

$$\deg_{\mathbf{w}} \phi\bigl(\eta_{i-1}(f_{i-1}, r)\bigr) = a_{i-1} \deg_{\mathbf{w}} \phi(r) = a_{i-1} \deg_{\mathbf{w}} \phi(\tilde{r}) = a_{i-1}\beta$$

by Lemma 5.4. Since $\deg_{\mathbf{w}} \phi(\eta_{i-1}(f_{i-1}, r)) = \phi(f_{i-2}f_i) = \epsilon_{i-2} + \epsilon_i$, this gives that $\epsilon_{i-2} = a_{i-1}\beta - \epsilon_i = a_{i-1}\beta - \delta_i$.

Now, we prove (i) and (ii) simultaneously by descending induction on l. When $l = i'$, we have only to check (i). Assume that $d(i-1) > 0$ and $l = i' = i$. Then, the first part of (i) is obvious. Since $a_i \geq 2$ by Lemma 3.1 (ii), and $v_2 \geq 1$, we have $a_i v_2 - 1 > 0$. Hence, we see from (9.4) that ϵ_{i-1} and $\epsilon_i = \delta_i$ are linearly independent by virtue of (a), proving the second part of (i). Thus, the statements hold. Assume that $d(i-1) < 0$ and $l = i' = i-1$. Then, we have $\epsilon_{i-2} = a_{i-1}\beta - \delta_i$ as mentioned. Since $t_{i-2} = t_i$ and $t_i a_{i-1} - a_{i-2} = a_i$ by (1.7), it follows that

$$\begin{aligned} d(l-1) = d(i-2) &= t_{i-2}\epsilon_{i-2} - a_{i-2}\beta = t_i(a_{i-1}\beta - \delta_i) - a_{i-2}\beta \\ &= (t_i a_{i-1} - a_{i-2})\beta - t_i\delta_i = a_i\beta - t_i\delta_i. \end{aligned}$$

Since $\beta \geq \alpha$ by (c) and $\alpha \geq \alpha'$, we know that

$$a_i\beta - t_i\delta_i \geq a_i\alpha' - t_i\delta_i > 0 \tag{9.5}$$

by (9.3). Hence, we get $d(l-1) > 0$. Since $\epsilon_{i-2} = a_{i-1}\beta - \delta_i$, it follows from (e) that $\epsilon_{l-1} = \epsilon_{i-2}$ and $\epsilon_l = \epsilon_{i-1}$ are linearly independent. Therefore, the statements are true.

Next, assume that $1 \leq l \leq i' - 1$. Then, we have $d(l) > 0$ by induction assumption. This implies $\epsilon_{l-1} = t_l\epsilon_l - \epsilon_{l+1}$ as remarked. Hence, we get the first part of (ii). Since ϵ_l and ϵ_{l+1} are linearly independent by induction assumption, it follows that ϵ_{l-1} and ϵ_l are linearly independent, proving the latter part of (i).

We show that $\epsilon_{l-1} > \epsilon_{l+1}$. Note that

$$\epsilon_{j-1} = t_j\epsilon_j - \epsilon_{j+1} \quad \text{for} \quad l \leq j \leq i' - 1, \tag{9.6}$$

where the case $j = l$ is just mentioned above, and the case $l < j \leq i' - 1$ is due to the induction assumption. Using this equality for $j = l$, we get

$$\epsilon_{l-1} - \epsilon_{l+1} = (t_l\epsilon_l - \epsilon_{l+1}) - \epsilon_{l+1} = t_l\epsilon_l - 2\epsilon_{l+1}. \tag{9.7}$$

First, consider the case where $l = i' - 1$. When $d(i-1) > 0$, we have $l = i - 1$. Since $a_i \geq 2$, $v_1 \geq 1$ and $v_2 \geq 1$, we know by (9.4) that $\epsilon_{i-1} > 2\delta_i = 2\epsilon_i$. Hence, we have $\epsilon_l > 2\epsilon_{l+1}$. Thus, we obtain $\epsilon_{l-1} > \epsilon_{l+1}$ from (9.7). When $d(i-1) < 0$, we have $l = i - 2$. Since $\epsilon_{i-2} = a_{i-1}\beta - \delta_i$ as remarked, it follows from (9.7) that

$$\epsilon_{l-1} - \epsilon_{l+1} = t_{i-2}\epsilon_{i-2} - 2\epsilon_{i-1} = t_i(a_{i-1}\beta - \delta_i) - 2\epsilon_{i-1}. \tag{9.8}$$

Since $\beta \geq \epsilon_{i-1} - v\delta_i$ by (e), the right-hand side of this equality is at least

$$t_i\bigl(a_{i-1}(\epsilon_{i-1} - v\delta_i) - \delta_i\bigr) - 2\epsilon_{i-1} = (t_i a_{i-1} - 2)\epsilon_{i-1} - t_i(a_{i-1}v + 1)\delta_i.$$

Note that (9.4) implies

$$\epsilon_{i-1} \geq a_i(v+1)\delta_i + \delta > a_i(v+1)\delta_i + a_i v\delta_i = a_i(2v+1)\delta_i,$$

since $v_1 \geq 1$, $a_i v_2 - 1 \geq 1$, and $\delta > a_i v\delta_i$ by (e). Hence, the right-hand side of the preceding equality is greater than

$$\begin{aligned} &\bigl((t_i a_{i-1} - 2)a_i(2v+1) - t_i(a_{i-1}v + 1)\bigr)\delta_i \\ &= \bigl(t_i a_{i-1}\bigl(a_i(2v+1) - v\bigr) - 2a_i(2v+1) - t_i\bigr)\delta_i \\ &= \bigl(a_{i-2}\bigl(a_i(2v+1) - v\bigr) + a_i\bigl((a_i - 2)(2v+1) - v\bigr) - t_i\bigr)\delta_i, \end{aligned} \tag{9.9}$$

where we use $t_i a_{i-1} = a_{i-2} + a_i$ for the last equality. When $a_i \geq 3$, we have $(a_i - 2)(2v+1) - v \geq 1$. Since $a_i > t_i$ by Lemma 3.1 (iii), and

$a_i(2v+1)-v>0$, we see that the right-hand side of (9.9) is positive. When $a_i \le 2$, we have $a_i = 2$ and $i = 3$ in view of (i) and (ii) of Lemma 3.1. Since $a_3 = t_1(t_0-1)-1$, and $t_0 \ge 3$ and $(t_0,t_1) \ne (3,1)$ by assumption, it follows that $(t_0,t_1) = (4,1)$. Hence, the right-hand side of (9.9) is equal to

$$\big((2(2v+1)-v) + 2(-v) - t_3\big)\delta_3 = (v+1)\delta_3 > 0.$$

Thus, we know that (9.8) is positive. Therefore, we get $\epsilon_{l-1} > \epsilon_{l+1}$.

Next, consider the case where $l = i'-2$. In this case, we have $\epsilon_l > \epsilon_{l+2}$ by the induction assumption of (ii). By (9.6), we have $\epsilon_{j-1} = t_j\epsilon_j - \epsilon_{j+1}$ for $j = l, l+1$. Hence, it follows from (9.7) that

$$\begin{aligned}\epsilon_{l-1} - \epsilon_{l+1} &= \frac{t_l}{2}\epsilon_l + \frac{t_l}{2}\epsilon_l - 2\epsilon_{l+1} = \frac{t_l}{2}\epsilon_l + \frac{t_l}{2}(t_{l+1}\epsilon_{l+1} - \epsilon_{l+2}) - 2\epsilon_{l+1}\\ &= \frac{t_l}{2}(\epsilon_l - \epsilon_{l+2}) + \frac{1}{2}(t_lt_{l+1}-4)\epsilon_{l+1} \ge \frac{t_l}{2}(\epsilon_l - \epsilon_{l+2}) > 0,\end{aligned}$$

since $t_lt_{l+1} = t_0t_1 \ge 4$. Therefore, we get $\epsilon_{l-1} > \epsilon_{l+1}$.

Finally, consider the case where $1 \le l \le i'-3$. Since $l+2 = i'-1$, we have $\epsilon_{j-1} = t_j\epsilon_j - \epsilon_{j+1}$ for $j = l, l+1, l+2$ by (9.6), and $\epsilon_{l+1} > \epsilon_{l+3}$ by the induction assumption of (ii). By Lemma 3.2, it follows that

$$\epsilon_{l-1} - \epsilon_{l+1} = (t_0t_1-4)\epsilon_{l+1} + (\epsilon_{l+1} - \epsilon_{l+3}) \ge \epsilon_{l+1} - \epsilon_{l+3} > 0.$$

Therefore, we get $\epsilon_{l-1} > \epsilon_{l+1}$. This proves the second part of (ii).

It remains only to show that $d(l-1) > 0$. Since $\epsilon_{l-1} = t_l\epsilon_l - \epsilon_{l+1}$ and $a_{l+1} = t_{l+1}a_l - a_{l-1}$, we have

$$\begin{aligned}d(l+1) + d(l-1) &= (t_{l+1}\epsilon_{l+1} - a_{l+1}\beta) + (t_{l-1}\epsilon_{l-1} - a_{l-1}\beta)\\ &= t_{l+1}(\epsilon_{l+1} + \epsilon_{l-1}) - (a_{l+1} + a_{l-1})\beta = t_{l+1}t_l\epsilon_l - t_{l+1}a_l\beta = t_{l+1}d(l).\end{aligned}$$

Since $d(l) > 0$ by induction assumption, it follows that $d(l-1) > -d(l+1)$. When $l = i'-1$ and $d(i-1) > 0$, we have

$$d(l+1) = d(i') = d(i) = t_i\epsilon_i - a_i\beta = t_i\delta_i - a_i\beta < 0$$

by (9.5). Hence, we get $d(l-1) > -d(l+1) > 0$. When $l = i'-1$ and $d(i-1) < 0$, we have $d(l+1) = d(i') = d(i-1) < 0$. Hence, we get $d(l-1) > 0$ similarly.

Assume that $1 \le l \le i'-2$. Then, we have $d(l'-1) > 0$ for $l' = i'-1, i'$ by induction assumption, and $l-1 \le i'-3 < i'-1$ if $l \equiv i' \pmod 2$, and $l-1 \le i'-4 < i'-2$ otherwise. Note that $\epsilon_{j-1} > \epsilon_{j+1}$ for $l \le j \le i'-1$, where the case $j = l$ is just verified above, and the case $l < j \le i'-1$ is due to the induction assumption. Hence, we know that $\epsilon_{l-1} > \epsilon_{i'-1}$ if $l \equiv i' \pmod 2$, and $\epsilon_{l-1} > \epsilon_{i'-2}$ otherwise. On the other hand, we have $a_0 = 1 < t_0 - 1 = a_2$, and $a_{j-1} < a_{j+1}$ for $j \ge 2$ by Lemma 3.1 (i). Hence, we similarly obtain that $a_{l-1} < a_{i'-1}$ if $l \equiv i' \pmod 2$, and $a_{l-1} < a_{i'-2}$ otherwise. Since $d(l'-1) > 0$ for $l' = i'-1, i'$, we have $a_{l'-1}\beta < t_{l'-1}\epsilon_{l'-1}$ for $l' = i'-1, i'$. Thus, it follows that

$$a_{l-1}\beta < a_{l'-1}\beta < t_{l'-1}\epsilon_{l'-1} = t_{l-1}\epsilon_{l'-1} < t_{l-1}\epsilon_{l-1},$$

where $l' := i'$ if $l \equiv i' \pmod 2$, and $l' := i'-1$ otherwise. Therefore, we get $d(l-1) = t_{l-1}\epsilon_{l-1} - a_{l-1}\beta > 0$. This proves that (i) and (ii) hold for every l. $\square$

10. Wildness (II)

Thanks to Theorem 1.1 (ii), the former case of Theorem 1.1 (iii) is reduced to the latter case. The latter case of Theorem 1.1 (iii) is divided into the following three cases:

(w1) $t_0 \geq 3$, $(t_0, t_1) \neq (3, 1)$ and $i \geq 3$.
(w2) $(t_0, t_1, i) = (3, 1, 3), (3, 1, 4)$.
(w3) $t_0 \geq 3$ and $i = 2$.

We show that the case (w3) is contained in Theorem 1.5 (ii). Let $\lambda(y) = y$ and $\mu(y, z) = \sum_{j=1}^{t_1} \alpha_j^1 z^j$. Then, we have

$$r_2 = \lambda(f_2)x_2 - \mu(f_2, f_1) = f_2 x_2 - \sum_{j=1}^{t_1} \alpha_j^1 x_1^j = r$$

by (1.2). Hence, we get

$$f_1 \tilde{f}_3 = \tilde{\eta}_2\left(f_2, r_2 f_2^{-1}\right) f_2^{a_2} = \tilde{\eta}_2\left(f_2, r f_2^{-1}\right) f_2^{a_2} = \left(f_2 + \sum_{j=1}^{t_0} \alpha_j^0 \left(r f_2^{-1}\right)^{j-1}\right) f_2^{t_0 - 1}$$

$$= f_2^{t_0} + \sum_{j=1}^{t_0} \alpha_j^0 r^{j-1} f_2^{t_0 - j} = \eta_2(f_2, r) = f_1 f_3.$$

Thus, it follows that $f_3 = \tilde{f}_3$. Therefore, we conclude that $D_2 = \tilde{D}_2$. Since $\lambda(y) = y$ does not belong to k, the wildness of $\exp hD_2$ follows from Theorem 1.5 (ii).

This section is devoted to proving Theorem 1.1 (iii) in the case of (w1), and Theorem 1.5 (ii). The case (w2) of Theorem 1.1 (iii) will be treated in Section 11.

First, we prove Theorem 1.1 (iii) in the case of (w1). Take any $h \in \ker D_i \setminus \{0\}$ and put $\phi = \exp hD_i$. By definition, we have $D_i = \Delta_{(f_{i+1}, f_i)}$. Since $i \geq 3$, we have

$$f_{i-1} f_{i+1} = q_i = \eta_i(f_i, r) = \tilde{\eta}_i(f_i, r).$$

Hence, D_i is obtained by the construction stated before Theorem 9.1 from the data $(g, s, \nu(y)) = (f_{i+1}, r, 1)$. Since $\ker D_i = k[f_i, f_{i+1}]$ by Theorem 1.1 (i), h belongs to $k[f_i, f_{i+1}] \setminus \{0\}$. Therefore, it suffices to check (a) through (e) by virtue of Theorem 9.1.

Since $i \geq 3$, we have $s = r = \tilde{r}$. Hence, we get $\alpha = \beta$, proving (c). Since δ_i and δ_{i+1} are linearly independent by Proposition 8.2 (ii), we get (a). Since $D_i(r) = f_i f_{i+1}$ by Theorem 1.1 (i), we have

$$\phi(r) = r + h f_i f_{i+1}.$$

We show that $\deg_{\mathbf{w}} r < \deg_{\mathbf{w}} h f_i f_{i+1}$. Since $t_0 \geq 3$ by assumption, $\deg_{\mathbf{w}} r = (1, 1, 1)$ is less than $(t_0, 0, t_0) = t_0 \delta_2$. By Proposition 8.1, we have $t_0 \delta_2 = \delta_1 + \delta_3$. Since $\delta_1 < \delta_2$ in view of (8.1), we get $\delta_1 + \delta_3 < \delta_2 + \delta_3$. By Proposition 8.2 (iv), we know that $\delta_j + \delta_{j+1} < \delta_{j+1} + \delta_{j+2}$ for each $j \geq 1$. Hence, it follows that $\delta_2 + \delta_3 < \delta_i + \delta_{i+1}$, since $i \geq 3$. Since $h \neq 0$, we have $\delta_i + \delta_{i+1} \leq \deg_{\mathbf{w}} h f_i f_{i+1}$. Thus, we conclude that $\deg_{\mathbf{w}} r < \deg_{\mathbf{w}} h f_i f_{i+1}$. Therefore, we get $\alpha = \deg_{\mathbf{w}} \phi(r) = \deg_{\mathbf{w}} h f_i f_{i+1}$.

Since $\deg_{\mathbf{w}} f_i = \delta_i$ and $\deg_{\mathbf{w}} f_{i+1} = \delta_{i+1}$ are linearly independent, we know that $f_i^{\mathbf{w}}$ and $f_{i+1}^{\mathbf{w}}$ are algebraically independent over k. Hence, we have $k[f_i, f_{i+1}]^{\mathbf{w}} = k[f_i^{\mathbf{w}}, f_{i+1}^{\mathbf{w}}]$ by the discussion before Lemma 1.1. Since h belongs to $k[f_i, f_{i+1}] \setminus \{0\}$, it follows that $\deg_{\mathbf{w}} h = v_1'\delta_i + v_2'\delta_{i+1}$ for some $v_1', v_2' \in \mathbf{Z}_{\geq 0}$. Thus, we get

$$\alpha = \deg_{\mathbf{w}} hf_if_{i+1} = (v_1' + 1)\delta_i + (v_2' + 1)\delta_{i+1}.$$

Because $v = \deg_y \nu(y) = 0$, we see that (b) holds for $v_j = v_j' + 1$ for $j = 1, 2$. Consequently, we have $\epsilon_{i-1} = a_i\alpha - \delta_{i+1}$ by (9.1). Since $t_{i+1}a_i = a_{i-1} + a_{i+1}$ by (1.7) and $\alpha > \delta_{i+1}$, it follows that

$$t_{i-1}\epsilon_{i-1} = t_{i+1}(a_i\alpha - \delta_{i+1}) = (a_{i-1} + a_{i+1})\alpha - t_{i+1}\delta_{i+1} > a_{i-1}\alpha + (a_{i+1} - t_{i+1})\delta_{i+1}.$$

Since $a_{i+1} > t_{i+1}$ by Lemma 3.1 (iii), the right-hand side of this inequality is greater than $a_{i-1}\alpha = a_{i-1}\beta$. Thus, we get $d(i-1) = t_{i-1}\epsilon_{i-1} - a_{i-1}\beta > 0$, proving (d) and (e). Therefore, (a) through (e) are fulfilled. This completes the proof of Theorem 1.1 (iii) in the case of (w1).

Next, we prove Theorem 1.5 (ii). First, we consider the case where $i = 2$ and $t_0 \geq 3$. Let J be the set of $j \geq 1$ such that $u_j := \deg_y \mu_j(y)$ is equal to $v := \deg_y \lambda(y)$, and let c and c_j be the leading coefficients of $\lambda(y)$ and $\mu_j(y)$ for each $j \in J$, respectively. Since $\mu(y, z)$ is an element of $zk[y, z]$, and $\lambda(y)$ and $\mu(y, z)$ have no common factor by assumption, we see that

$$\bar{\mu}(y, z) := \mu(y, z) - c^{-1}\lambda(y)\sum_{j\in J} c_j z^j$$

belongs to $zk[y, z]$, and $\lambda(y)$ and $\bar{\mu}(y, z)$ have no common factor. We show that $\bar{\mu}(y, z) = 0$ if and only if $\lambda(y)$ belongs to $k^\times$ and $\mu(y, z)$ belongs to $zk[z]$. If $\bar{\mu}(y, z) = 0$, then we have $\mu(y, z) = c^{-1}\lambda(y)\sum_{j\in J} c_j z^j$. Since $\lambda(y)$ and $\mu(y, z)$ have no common factor, it follows that $\lambda(y)$ belongs to $k^\times$, and so $\mu(y, z)$ belongs to $zk[z]$. Conversely, if $\lambda(y)$ belongs to $k^\times$ and $\mu(y, z)$ belongs to $zk[z]$, then we have $\lambda(y) = c$ and $\mu(y, z) = \sum_{j\in J} c_j z^j$. Hence, we get $\bar{\mu}(y, z) = 0$.

Define $\tau \in J(k[x_1]; x_2, x_3)$ by

$$\tau(x_2) = x_2 + c^{-1}\sum_{j\in J} c_j x_1^j \quad \text{and} \quad \tau(x_3) = x_3 + x_1^{-1}\bigl(\theta(\tau(x_2)) - \theta(x_2)\bigr).$$

Then, we have $\tau(f_1) = \tau(x_1) = x_1 = f_1$,

$$\tau(f_2) = x_1\tau(x_3) - \theta(\tau(x_2)) = x_1x_3 - \theta(x_2) = f_2$$

and

$$\tau(r_2) = \lambda(f_2)\tau(x_2) - \mu(f_2, x_1) = \lambda(f_2)x_2 - \bar{\mu}(f_2, x_1).$$

Hence, $\tau(\tilde{f}_3)$ is equal to the polynomial obtained similarly to $\tilde{f}_3$ from $\lambda(y)$ and $\bar{\mu}(y, z)$ instead of $\lambda(y)$ and $\mu(y, z)$. By the formula (2.1), we get

$$D' := \Delta_{(\tau(\tilde{f}_3), f_2)} = \Delta_{(\tau(\tilde{f}_3), \tau(f_2))} = \tau \circ \tilde{D}_2 \circ \tau^{-1}, \tag{10.1}$$

since $\det J\tau = 1$.

Now, take any $h \in \ker \tilde{D}_2 \setminus \{0\}$. Assume that $\lambda(y)$ belongs to $k^\times$ and $\mu(y, z)$ belongs to $zk[z] \setminus \{0\}$. Then, we show that $\exp h\tilde{D}_2$ is tame if and only if h belongs to $k[\tilde{f}_3]$. By assumption, it follows that $\lambda(y) = c$ and $\bar{\mu}(y, z) = 0$. Hence, we have $\tau(\tilde{f}_3) = c^{a_2}x_3$ by (1.8). Since f_2 is a symmetric

polynomial in x_1 and x_3 over $k[x_2]$, we may regard $D' = c^{a_2}\Delta_{(x_3,f_2)}$ as $c^{a_2}\Delta_{(x_1,f_2)} = -c^{a_2}D_1$ by interchanging x_1 and x_3. Put $h' = \tau(h)$. Then, h' belongs to $\ker D' \setminus \{0\}$ by (10.1). Note that $\exp h\tilde{D}_2$ is tame if and only if $\exp h'D'$ is tame. By the discussion after Theorem 1.1, it follows that $\exp h'D'$ is tame if and only if h' belongs to $k[x_3]$ or $t_0 \leq 2$, and hence if and only if h' belongs to $k[x_3]$ by the assumption that $t_0 \geq 3$. Since $k[x_3] = k[\tau(\tilde{f}_3)]$, we conclude that $\exp h\tilde{D}_2$ is tame if and only if h belongs to $k[\tilde{f}_3]$. When this is the case, $h'D'$ is triangular if x_1 and x_3 are interchanged. Hence, $h\tilde{D}_2$ is tamely triangularizable, proving the last part of Theorem 1.5 (ii).

Take any $h \in \ker \tilde{D}_i \setminus \{0\}$. To complete the proof of Theorem 1.5 (ii), it suffices to prove that $\exp h\tilde{D}_i$ is wild in the following cases:
(1) $i = 2$, $t_0 \geq 3$ and $\bar{\mu}(y,z) \neq 0$.
(2) $i \geq 3$, $t_0 \geq 3$ and $(t_0, t_1) \neq (3,1)$.

In the case of (1), we may replace $\mu(y,z)$ with $\bar{\mu}(y,z)$ because of (10.1). Therefore, we may assume that $u_j \neq v$ for each $j \geq 1$, since $u_j \neq v$ if j does not belong to J, and $\deg_y(\mu_j(y) - c^{-1}c_j\lambda(y)) < v$ otherwise.

By definition, we have $\tilde{D}_i = \Delta_{(\tilde{f}_{i+1},f_i)}$, and

$$f_{i-1}\tilde{f}_{i+1} = \tilde{q}_i = \tilde{\eta}_i(f_i, r_i')\lambda(f_i)^{a_i},$$

where $r_i' := r_i\lambda(f_i)^{-1}$. Hence, $\tilde{D}_i$ is obtained by the construction stated before Theorem 9.1 from the data $(g, s, \nu(y)) = (\tilde{f}_{i+1}, r_i, \lambda(y))$. Since $\ker \tilde{D}_i = k[f_i, \tilde{f}_{i+1}]$ by Theorem 1.5 (i), h belongs to $k[f_i, \tilde{f}_{i+1}] \setminus \{0\}$. Therefore, it suffices to check (a) through (e) by virtue of Theorem 9.1.

Since $\mu(y,z) \neq 0$ by assumption, $J' := \{j \geq 1 \mid \mu_j(y) \neq 0\}$ is not empty. Since δ_{i-1} and δ_i are linearly independent by Proposition 8.2 (ii), we see that $\deg_{\mathbf{w}} \mu_j(f_i)f_{i-1}^j = u_j\delta_i + j\delta_{i-1}$'s are different for different elements j's of J'. Hence, we may find $l_1 \in J'$ such that

$$\deg_{\mathbf{w}} \mu(f_i, f_{i-1}) = \deg_{\mathbf{w}} \mu_{l_1}(f_i)f_{i-1}^{l_1} = u_{l_1}\delta_i + l_1\delta_{i-1}.$$

By Proposition 8.2 (v), the second components of δ_{i-1} and δ_i are zero, while $\deg_{\mathbf{w}} \tilde{r}$ equals $(0,1,0)$ if $i = 2$, and $(1,1,1)$ if $i \geq 3$. Because δ_{i-1} and δ_i are linearly independent, it follows that δ_{i-1}, δ_i and $\deg_{\mathbf{w}} \tilde{r}$ are linearly independent. Hence, $\deg_{\mathbf{w}} \mu(f_i, f_{i-1})$ is not equal to $\deg_{\mathbf{w}} \lambda(f_i)\tilde{r} = v\delta_i + \deg_{\mathbf{w}} \tilde{r}$. Thus, we know that

(10.2)
$$\deg_{\mathbf{w}} r_i = \deg_{\mathbf{w}}\bigl(\lambda(f_i)\tilde{r} + \mu(f_i, f_{i-1})\bigr) = \max\{v\delta_i + \deg_{\mathbf{w}} \tilde{r}, u_{l_1}\delta_i + l_1\delta_{i-1}\},$$

and so

$$\deg_{\mathbf{w}} r_i' = \deg_{\mathbf{w}} r_i - v\delta_i \geq \deg_{\mathbf{w}} \tilde{r} > 0.$$

Since δ_{i-1}, δ_i and $\deg_{\mathbf{w}} \tilde{r}$ are linearly independent, and $l_1 \geq 1$, we see from (10.2) that δ_i and $\deg_{\mathbf{w}} r_i$ are linearly independent. Therefore, δ_i and $\deg_{\mathbf{w}} r_i'$ are linearly independent. In the case of (1), it follows that

$$\deg_{\mathbf{w}} \tilde{\eta}_2(f_2, r_2') = \deg_{\mathbf{w}} \bigl(f_2 + \theta(r_2')\bigr) = \max\{\delta_2, a_2 \deg_{\mathbf{w}} r_2'\},$$

since $\deg_z \theta(z) = t_0 - 1 = a_2$. In the case of (2), we have $t_i\delta_i \neq a_i \deg_{\mathbf{w}} r_i'$. Hence, we get

$$\deg_{\mathbf{w}} \tilde{\eta}_i(f_i, r_i') = \deg_{\mathbf{w}} \eta_i(f_i, r_i') = \max\{t_i\delta_i, a_i \deg_{\mathbf{w}} r_i'\}$$

by applying Lemma 5.4 with $f = f_i$ and $p = r'_i$. Set $\tilde{t}_2 = 1$ in the case of (1), and $\tilde{t}_i = t_i$ in the case of (2). Then, we have

$$\begin{aligned} \delta := \deg_{\mathbf{w}} \tilde{f}_{i+1} &= \deg_{\mathbf{w}} \tilde{\eta}_i(f_i, r'_i)\lambda(f_i)^{a_i} f_{i-1}^{-1} \\ &= \max\{\tilde{t}_i\delta_i, a_i \deg_{\mathbf{w}} r'_i\} + a_i v\delta_i - \delta_{i-1} \\ &= \max\{(\tilde{t}_i + a_i v)\delta_i, a_i \deg_{\mathbf{w}} r_i\} - \delta_{i-1}. \end{aligned} \tag{10.3}$$

Now, let us prove (a). From (10.3) and (10.2), we see that δ and r_i have two possibilities. In the case where $\delta = a_i \deg_{\mathbf{w}} r_i - \delta_{i-1}$ and $\deg_{\mathbf{w}} r_i = u_{l_1}\delta_i + l_1\delta_{i-1}$, we have

$$\delta = a_i u_{l_1}\delta_i + (a_i l_1 - 1)\delta_{i-1}.$$

Note that $a_2 = t_0 - 1 \geq 2$ in the case of (1), and $a_i \geq 2$ in the case of (2) by Lemma 3.1 (ii). Hence, we have $a_i l_1 - 1 \geq a_i - 1 \geq 1$. Thus, we know that δ_i and δ are linearly independent, since so are δ_i and δ_{i-1}. In the other cases, we can easily check that δ_i and δ are linearly independent because δ_{i-1}, δ_i and $\deg_{\mathbf{w}} \tilde{r}$ are linearly independent. Therefore, we get (a).

Recall that $u_j \neq v$ for each j in the case of (1). Hence, we have $(u_{l_1}, v) \neq (0, 0)$. From (10.2), we see that $\deg_{\mathbf{w}} r_2 > \delta_2$. Since $a_2 \geq 2$, and $\delta_1 = (1, 0, 0)$ is less than $\delta_2 = (1, 0, 1)$, it follows from (10.3) that

$$\delta \geq a_2 \deg_{\mathbf{w}} r_2 - \delta_1 \geq \deg_{\mathbf{w}} r_2 + (\deg_{\mathbf{w}} r_2 - \delta_2) + (\delta_2 - \delta_1) > \deg_{\mathbf{w}} r_2 > \delta_2.$$

Therefore, we get the second part of (b) for $i = 2$. Since $\tilde{D}_2(r_2) = \lambda(f_2)\tilde{f}_3$ by Theorem 1.5 (i), we have $\phi(r_2) = r_2 + h\lambda(f_2)\tilde{f}_3$. By the preceding inequality, we know that $\deg_{\mathbf{w}} h\lambda(f_2)\tilde{f}_3 \geq \delta > \deg_{\mathbf{w}} r_2$. Hence, we get

$$\alpha = \deg_{\mathbf{w}} \phi(r_2) = \deg_{\mathbf{w}} h\lambda(f_2)\tilde{f}_3 \geq \deg_{\mathbf{w}} \lambda(f_2)\tilde{f}_3 = v\delta_2 + \delta,$$

proving the first part of (b) for $i = 2$.

In the case of (2), it follows from (10.3) that

$$\delta \geq (t_i + a_i v)\delta_i - \delta_{i-1} = (t_i\delta_i - \delta_{i-1}) + a_i v\delta_i = \delta_{i+1} + a_i v\delta_i > a_i v\delta_i,$$

since $t_i\delta_i - \delta_{i-1} = \delta_{i+1}$ by Proposition 8.1. This proves the second part of (e).

Since $\deg_{\mathbf{w}} f_i = \delta_i$ and $\deg_{\mathbf{w}} \tilde{f}_{i+1} = \delta$ are linearly independent by (a), we have $k[f_i, \tilde{f}_{i+1}]^{\mathbf{w}} = k[f_i^{\mathbf{w}}, \tilde{f}_{i+1}^{\mathbf{w}}]$. Hence, we may write

$$\deg_{\mathbf{w}} h = v'_1\delta_i + v'_2\delta,$$

where $v'_1, v'_2 \in \mathbf{Z}_{\geq 0}$. In the case of (1), we have $\alpha = \deg_{\mathbf{w}} h\lambda(f_2)\tilde{f}_3$ as mentioned. Hence, we get

$$\alpha = \deg_{\mathbf{w}} h\lambda(f_2)\tilde{f}_3 = (v'_1 + v)\delta_2 + (v'_2 + 1)\delta. \tag{10.4}$$

Consider the case of (2). Since $\tilde{D}_i(r_i) = \lambda(f_i)f_i\tilde{f}_{i+1}$ by Theorem 1.5 (i), we have

$$\phi(r_i) = r_i + h\lambda(f_i)f_i\tilde{f}_{i+1}.$$

We show that $(v + 1)\delta_i + \delta$ is greater than $\deg_{\mathbf{w}} r_i$. Then, it follows that

$$\deg_{\mathbf{w}} h\lambda(f_i)f_i\tilde{f}_{i+1} \geq (v + 1)\delta_i + \delta > \deg_{\mathbf{w}} r_i.$$

Consequently, we obtain

$$\alpha = \deg_{\mathbf{w}} \phi(r_i) = \deg_{\mathbf{w}} h\lambda(f_i)f_i\tilde{f}_{i+1} = (v'_1 + v + 1)\delta_i + (v'_2 + 1)\delta, \tag{10.5}$$

and thereby proving that (b) for $i \geq 3$ holds with $v_1 = v_1'+1$ and $v_2 = v_2'+1$. First, assume that $a_i \deg_{\mathbf{w}} r_i' < t_i\delta_i$. Then, we have

$$\deg_{\mathbf{w}} r_i = \deg_{\mathbf{w}} r_i' + v\delta_i < \left(\frac{t_i}{a_i} + v\right)\delta_i < (1+v)\delta_i < (v+1)\delta_i + \delta,$$

since $t_i < a_i$ by Lemma 3.1 (iii). Hence, the assertion holds. Next, assume that $a_i \deg_{\mathbf{w}} r_i' \geq t_i\delta_i$. Since $\deg_{\mathbf{w}} r_i'$ and δ_i are linearly independent, it follows that $a_i \deg_{\mathbf{w}} r_i' > t_i\delta_i$, and so $a_i \deg_{\mathbf{w}} r_i > (t_i + a_i v)\delta_i$. Hence, we know that $\delta = a_i \deg_{\mathbf{w}} r_i - \delta_{i-1}$ by (10.3). Thus, we have

$$\begin{aligned}
(v+1)\delta_i + \delta - \deg_{\mathbf{w}} r_i &= (v+1)\delta_i + (a_i - 1)\deg_{\mathbf{w}} r_i - \delta_{i-1}\\
&> \left(v + 1 + (a_i-1)\frac{t_i + a_i v}{a_i}\right)\delta_i - \delta_{i-1}\\
&= \left(1 + a_i v - \frac{t_i}{a_i}\right)\delta_i + t_i\delta_i - \delta_{i-1} > t_i\delta_i - \delta_{i-1} = \delta_{i+1} > 0.
\end{aligned}$$

Therefore, we get $(v+1)\delta_i + \delta > \deg_{\mathbf{w}} r_i$. This proves (10.5), and (b) for $i \geq 3$.

Recall that (b) implies (9.1). By (10.4) and (10.5), it follows that $\epsilon_{i-1} = a_i\alpha - \delta$ and δ_i are linearly independent, since δ and δ_i are linearly independent, and $a_i(v_2'+1) - 1 \geq 1$. Hence, $\deg_{\mathbf{w}} \mu_j(f_i)\phi(f_{i-1})^j = u_j\delta_i + j\epsilon_{i-1}$'s are different for different elements j's of J'. Thus, we may find $l_2 \in J'$ such that

$$\gamma := \deg_{\mathbf{w}} \mu\big(f_i, \phi(f_{i-1})\big) = \deg_{\mathbf{w}} \mu_{l_2}(f_i)\phi(f_{i-1})^{l_2} = u_{l_2}\delta_i + l_2\epsilon_{i-1}.$$

We show that $\gamma > \alpha$. If $\alpha > \delta$, then we have

$$\gamma \geq l_2\epsilon_{i-1} \geq \epsilon_{i-1} = a_i\alpha - \delta > (a_i - 1)\alpha \geq \alpha.$$

In particular, we have $\gamma > \alpha$ in the case of (2), since $\alpha > \delta$ by (10.5). Assume that $\alpha \leq \delta$ in the case of (1). Then, we see from (10.4) that $v_1' = v_2' = v = 0$ and $\alpha = \delta$. Since $u_{l_2} \neq v$ by assumption, it follows that $u_{l_2} > 0$. Hence, we get $\gamma \geq \delta_2 + \epsilon_1$. Since $\alpha = \delta$ and $a_2 \geq 2$, we have $\epsilon_1 = a_2\alpha - \delta \geq \alpha$. Thus, we conclude that $\gamma > \alpha$. Therefore, $\deg_{\mathbf{w}} \mu\big(f_i, \phi(f_{i-1})\big) = \gamma$ is greater than $\deg_{\mathbf{w}} \phi(r_i) = \alpha$. On the other hand, we have

$$\phi(r_i) = \phi\big(\lambda(f_i)\tilde{r} - \mu(f_i, f_{i-1})\big) = \lambda(f_i)\phi(\tilde{r}) - \mu\big(f_i, \phi(f_{i-1})\big), \tag{10.6}$$

since $\phi(f_i) = f_i$. This implies that $\deg_{\mathbf{w}} \lambda(f_i)\phi(\tilde{r}) = \deg_{\mathbf{w}} \mu\big(f_i, \phi(f_{i-1})\big) = \gamma$. Hence, we get

$$\beta = \deg_{\mathbf{w}} \phi(\tilde{r}) = \gamma - \deg_{\mathbf{w}} \lambda(f_i) = (u_{l_2} - v)\delta_i + l_2\epsilon_{i-1}. \tag{10.7}$$

From this, we know that $\beta \geq \epsilon_{i-1} - v\delta_i$. This proves the first part of (e). In the case of (2), we have $a_{i-1} \geq 2$ by Lemma 3.1 (ii), since $i - 1 \geq 2$. Hence, we get $a_{i-1}(u_{l_2} - v) - 1 \neq 0$. Because

$$a_{i-1}\beta - \delta_i = \big(a_{i-1}(u_{l_2} - v) - 1\big)\delta_i + a_{i-1}l_2\epsilon_{i-1}$$

by (10.7), and δ_i and ϵ_{i-1} are linearly independent as mentioned, it follows that $a_{i-1}\beta - \delta_i$ and ϵ_{i-1} are linearly independent, proving the last part of (e).

In the case of (1), we have $u_{l_2} \neq v$. Since δ_2 and ϵ_1 are linearly independent, we know by (10.7) that β and ϵ_1 are linearly independent, proving (c) for $i = 2$. In the case of (2), it follows from (10.7), (9.1) and (10.5) that

$$\begin{aligned}\beta - \alpha &\geq (\epsilon_{i-1} - v\delta_i) - \alpha = (a_i - 1)\alpha - \delta - v\delta_i \geq \alpha - \delta - v\delta_i \\ &\geq \big((v+1)\delta_i + \delta\big) - \delta - v\delta_i = \delta_i > 0.\end{aligned}$$

This proves (c) for $i \geq 3$.

Finally, we prove (d) by contradiction. Suppose that $d(i-1) = 0$. Then, we have

$$t_{i-1}\epsilon_{i-1} = a_{i-1}\beta = a_{i-1}\big((u_{l_2} - v)\delta_i + l_2\epsilon_{i-1}\big)$$

by (10.7). Since ϵ_{i-1} and δ_i are linearly independent, it follows that $t_{i-1} = a_{i-1}l_2$. Since $i - 1 \geq 2$, this contradicts Lemma 3.1 (v), proving (d). Thus, we have verified (a) through (e). Therefore, ϕ is wild in the case of (1) and (2) by Theorem 9.1. This completes the proof of Theorem 1.5 (ii).

11. Exceptional case

The goal of this section is to complete the proof of Theorem 1.1 by proving (a) of (i) in the case of $(t_0, t_1, i) = (3, 1, 4)$, and (ii) in the case of (w2). Assume that $(t_0, t_1) = (3, 1)$. Then, we have $b_2 = t_1b_1 - b_0 + \xi_1 = 1$, $b_3 = t_2b_2 - b_1 + \xi_2 = 2$ and $b_4 = t_3b_3 - b_2 + \xi_3 = 0$, since $b_0 = b_1 = 0$. Hence, we see from (6.9) that

$$\begin{aligned}&f_0f_2 = q_1 = r + f_1\\ &f_1f_3 = q_2 = f_2^3r^{-1}\big(r + \theta_0(f_2^{-1}r)\big) = f_2^3 + r^2 + \alpha_2^0f_2r + \alpha_1^0f_2^2\\ &f_2f_4 = q_3 = f_3r^{-1}\big(r + f_3^{-1}r^2\big) = f_3 + r\\ &f_3f_5 = q_4 = r + \theta(f_4)f_4,\end{aligned} \tag{11.1}$$

since $\theta(z) = z^2 + \alpha_2^0z + \alpha_1^0$, $\theta_0(z) = \theta(z)z$ and $\theta_1(z) = z$. By (1.2), we have $r = x_2f_2 - x_1$. Hence, the second equality of (11.1) gives that

$$\begin{aligned}f_1f_3 &= f_2^3 + (x_2f_2 - x_1)^2 + \alpha_2^0(x_2f_2 - x_1)f_2 + \alpha_1^0f_2^2\\ &= x_1^2 - x_1(2x_2 + \alpha_2^0)f_2 + (f_2 + x_2^2 + \alpha_2^0x_2 + \alpha_1^0)f_2^2.\end{aligned}$$

Since $f_1 = x_1$ and $f_2 + x_2^2 + \alpha_2^0x_2 + \alpha_1^0 = f_2 + \theta(x_2) = x_1x_3$ by (1.2), it follows that

$$f_3 = x_1 - (2x_2 + \alpha_2^0)f_2 + x_3f_2^2. \tag{11.2}$$

Thus, we know by the third equality of (11.1) that

$$f_2f_4 = r + f_3 = (x_2f_2 - x_1) + (x_1 - (2x_2 + \alpha_2^0)f_2 + x_3f_2^2) = f_2(x_3f_2 - x_2 - \alpha_2^0).$$

Therefore, we get

$$f_4 = x_3f_2 - x_2 - \alpha_2^0.$$

With the notation of Chapter 6, define $\tilde{D} = D_{-\theta}$. Then, we have $\tilde{D}(f_2) = 0$, since $f_2 = x_1x_3 - \theta(x_2) = f_{-\theta}$. Hence, $-f_2\tilde{D}$ belongs to $\mathrm{LND}_k\, k[\mathbf{x}]$. Set $\tilde{\sigma} = \exp(-f_2\tilde{D})$ and $y_i = \tilde{\sigma}(x_i)$ for $i = 1, 2, 3$. Then, we have

$$y_3 = x_3 \quad \text{and} \quad y_2 = x_2 - f_2x_3 = -f_4 - \alpha_2^0, \tag{11.3}$$

since $\tilde{D}(x_3) = 0$, and $\tilde{D}(x_2) = x_3$ and $\tilde{D}(f_2) = 0$. Since f_2 is fixed under $\tilde{\sigma}$, we see that $f_2 = x_1x_3 - \theta(x_2)$ is equal to $\tilde{\sigma}(f_2) = y_1x_3 - \theta(x_2 - f_2x_3)$. From this, we obtain

$$y_1 = x_1 + \big(\theta(x_2 - f_2x_3) - \theta(x_2)\big)x_3^{-1}.$$

Hence, we get

$$y_1 = x_1 - \theta'(x_2)f_2 + \frac{1}{2}\theta''(x_2)f_2^2x_3 = f_3, \tag{11.4}$$

since $\theta'(x_2) = 2x_2 + \alpha_2^0$ and $\theta''(x_2) = 2$. Thus, we can define $\sigma_3 \in \mathrm{Aut}(k[\mathbf{x}]/k)$ by

$$\sigma_3(x_1) = y_1 = f_3, \quad \sigma_3(x_2) = -y_2 - \alpha_2^0 = f_4, \quad \sigma_3(x_3) = y_3 = x_3. \tag{11.5}$$

Note that the sum of the two roots of $\theta(z)$ is equal to $-\alpha_2^0$. Hence, $\theta(z)$ and $\theta(-z-\alpha_2^0)$ have exactly the same roots. Since $\theta(z)$ and $\theta(-z-\alpha_2^0)$ are monic polynomials, we conclude that $\theta(-z-\alpha_2^0) = \theta(z)$. Thus, we have

$$\sigma_3(f_2) = y_1y_3 - \theta(-y_2 - \alpha_2^0) = y_1y_3 - \theta(y_2) = \tilde{\sigma}(f_2) = f_2.$$

Therefore, σ_3 belongs to $\mathrm{Aut}(k[\mathbf{x}]/k[f_2, x_3])$. Since $x_1x_3 = f_2 + \theta(x_2)$ by (1.2), we know that

$$f_3x_3 = \sigma_3(x_1x_3) = \sigma_3(f_2 + \theta(x_2)) = f_2 + \theta(f_4).$$

Hence, it follows that

$$\begin{aligned} f_3(x_3f_4 - 1 - f_5) &= (f_3x_3)f_4 - (f_3 + r) - (f_3f_5 - r) \\ &= (f_2 + \theta(f_4))f_4 - f_2f_4 - \theta(f_4)f_4 = 0 \end{aligned}$$

by the last two equalities of (11.1). Therefore, we get $f_5 = x_3f_4 - 1$.

Now, let us prove (a) of Theorem 1.1 (i) in the case of $(t_0, t_1, i) = (3, 1, 4)$. Since

$$D_4 = \Delta_{(f_5, f_4)} = \Delta_{(x_3f_4 - 1, f_4)} = f_4\Delta_{(x_3, f_4)}, \tag{11.6}$$

we see that D_4 is not irreducible, and x_3 belongs to $\ker D_4$. By (11.5), we have $\sigma_3^{-1}(x_3) = x_3$, $\sigma_3^{-1}(f_4) = x_2$ and $\sigma_3^{-1}(f_5) = \sigma_3^{-1}(x_3f_4 - 1) = x_2x_3 - 1$. Hence, we know that $\sigma_3^{-1}(x_3) = x_3$ does not belong to $\sigma_3^{-1}(k[f_4, f_5]) = k[x_2, x_2x_3 - 1]$. Thus, x_3 does not belong to $k[f_4, f_5]$. Therefore, we conclude that $\ker D_4 \neq k[f_4, f_5]$. This proves (a) of Theorem 1.1 (i) when $(t_0, t_1, i) = (3, 1, 4)$.

Next, we prove Theorem 1.1 (iii) in the case of $(t_0, t_1, i) = (3, 1, 4)$. Take any $h \in \ker D_4 \setminus \{0\}$. We show that $\exp hD_4 = \exp hf_4\Delta_{(x_3, f_4)}$ is wild. Since

$$\begin{aligned} \Delta_{(x_3, f_4)}(x_1) &= -\frac{\partial f_4}{\partial x_2} = 1 - x_3\frac{\partial f_2}{\partial x_2} = 1 + \theta'(x_2)x_3 \\ \Delta_{(x_3, f_4)}(x_2) &= \frac{\partial f_4}{\partial x_1} = x_3\frac{\partial f_2}{\partial x_1} = x_3^2 \quad \text{and} \quad \Delta_{(x_3, f_4)}(x_3) = 0, \end{aligned}$$

we see that $\Delta_{(x_3, f_4)}$ is triangular if x_1 and x_3 are interchanged. Since hf_4 belongs to $\ker \Delta_{(x_3, f_4)} \setminus k[x_3]$, and $\partial(\Delta_{(x_3, f_4)}(x_1))/\partial x_2 = \theta''(x_2)x_3 = 2x_3$ is not divisible by $\Delta_{(x_3, f_4)}(x_2) = x_3^2$, we conclude from Theorem 2.3 that $\exp hf_4\Delta_{(x_3, f_4)}$ is wild. Therefore, Theorem 1.1 (iii) is true when $(t_0, t_1, i) = (3, 1, 4)$.

The rest of this section is devoted to proving Theorem 1.1 (iii) in the case of $(t_0, t_1, i) = (3, 1, 3)$. Take any $h \in \ker D_3 \setminus \{0\}$ and put $\phi = \exp hD_3$. Since $q_3 = r + f_3$ by (11.1), we have $D_3(f_2) = (\partial q_3/\partial r)f_3 = f_3$ by (6.5) with $j = 3$. Hence, we get

$$\phi(f_2) = f_2 + hf_3.$$

First, we describe $z_i := \phi(x_i)$ for $i = 1, 2, 3$. By (11.3) and (11.4), we have

$$D_3 = \Delta_{(f_4,f_3)} = \Delta_{(-y_2-\alpha_2^0,y_1)} = \Delta_{(y_1,y_2)}.$$

Since $x_3 = y_3$, it follows that $D_3(x_3) = \Delta_{(y_1,y_2)}(y_3) = \det J\tilde{\sigma}$.

Here, we remark that $\det J(\exp D) = 1$ for any $D \in \mathrm{LND}_k\, k[\mathbf{x}]$. This is verified as follows. Let $R = k[t]$ be the polynomial ring in one variable over k. Then, D naturally extends to an element $\bar{D}$ of $\mathrm{LND}_R\, R[\mathbf{x}]$. Since $\Psi := \exp t\bar{D}$ belongs to $\mathrm{Aut}(R[\mathbf{x}]/R)$, we know that $\det J\Psi$ belongs to $R^\times = k^\times$. Put $R_0 = R/(t)$ and $R_1 = R/(t-1)$, and define an automorphism of $R_i \otimes_R R[\mathbf{x}] = R_i[\mathbf{x}] = k[\mathbf{x}]$ over $R_i = k$ by $\psi_i = \mathrm{id}_{R_i} \otimes \Psi$ for $i = 0, 1$. Then, we have $\psi_0 = \mathrm{id}_{k[\mathbf{x}]}$ and $\psi_1 = \exp D$. Hence, the images of $\det J\Psi$ in $R_0[\mathbf{x}]$ and $R_1[\mathbf{x}]$ are $\det J(\mathrm{id}_{k[\mathbf{x}]}) = 1$ and $\det J(\exp D)$, respectively. Since $\det J\Psi$ belongs to $k^\times$, it follows that $\det J(\exp D) = \det J\Psi = 1$.

By the remark, we know that $\det J\tilde{\sigma} = \det J(\exp(-f_2\tilde{D})) = 1$. Thus, we conclude that $D_3(x_3) = 1$. Therefore, we get

$$z_3 = x_3 + h.$$

Since $D_3 = \Delta_{(y_1,y_2)}$ kills y_1 and y_2, we have $\phi(y_i) = y_i$ for $i = 1, 2$. Hence, it follows from (11.3) that

$$y_2 = \phi(y_2) = \phi(x_2 - f_2x_3) = z_2 - \phi(f_2)z_3.$$

Thus, we get

$$z_2 = y_2 + \phi(f_2)z_3 = y_2 + (f_2 + hy_1)(x_3 + h). \tag{11.7}$$

Since $f_2 = x_1x_3 - \theta(x_2)$ is equal to $\tilde{\sigma}(f_2) = y_1x_3 - \theta(y_2)$, we know that $\phi(f_2) = z_1z_3 - \theta(z_2)$ is equal to $\phi(\tilde{\sigma}(f_2)) = y_1z_3 - \theta(y_2)$. Therefore, we have

$$\begin{aligned} z_1 &= y_1 + \big(\theta(z_2) - \theta(y_2)\big)z_3^{-1} \\ &= y_1 + \Big(\theta\big(y_2 + \phi(f_2)z_3\big) - \theta(y_2)\Big)z_3^{-1} \\ &= y_1 + (2y_2 + \alpha_2^0)\phi(f_2) + \phi(f_2)^2 z_3 \\ &= y_1 + (2y_2 + \alpha_2^0)(f_2 + hy_1) + (f_2 + hy_1)^2(x_3 + h). \end{aligned} \tag{11.8}$$

We show that y_2 is a W-test polynomial. Take any totally ordered additive group Λ and $\mathbf{v} \in (\Lambda_{>0})^3$. Then, $f_2^{\mathbf{v}}$ is equal to x_1x_3 or $-x_2^{t_0-1}$ or $x_1x_3 - x_2^{t_0-1}$ as mentioned before Proposition 9.2, since $t_0 = 3$. Hence, we have

$$y_2^{\mathbf{v}} = (x_2 - f_2x_3)^{\mathbf{v}} = -f_2^{\mathbf{v}}x_3,$$

and is equal to $-x_1x_3^2$ or $x_2^{t_0-1}x_3$ or $-(x_1x_3 - x_2^{t_0-1})x_3$. Thus, we know that $y_2^{\mathbf{v}}$ is not divisible by $x_i - g$ for any $i \in \{1, 2, 3\}$ and $g \in k[\mathbf{x} \setminus \{x_i\}] \setminus k$, and by $x_i^{s_i} - cx_j^{s_j}$ for any $i, j \in \{1, 2, 3\}$ with $i \neq j$, $s_i, s_j \in \mathbf{N}$ and $c \in k^\times$. Therefore, y_2 is a W-test polynomial due to Proposition 1.4.

Recall that $\Gamma = \mathbf{Z}^3$ has the lexicographic order with $\mathbf{e}_1 < \mathbf{e}_2 < \mathbf{e}_3$, and $\mathbf{w} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. Hence, we have $f_2^{\mathbf{w}} = x_1x_3$. By (11.4), (11.2) and (11.3), it follows that

$$y_1^{\mathbf{w}} = f_3^{\mathbf{w}} = (x_3f_2^2)^{\mathbf{w}} = x_1^2x_3^3 \quad \text{and} \quad y_2^{\mathbf{w}} = (-f_2x_3)^{\mathbf{w}} = -x_1x_3^2. \tag{11.9}$$

Hence, we get $\deg_{\mathbf{w}} y_1 > \deg_{\mathbf{w}} y_2 > \deg_{\mathbf{w}} f_2$. In view of (11.7) and (11.8), we see that

$$z_1^{\mathbf{w}} = (h^2y_1^2)^{\mathbf{w}}(x_3+h)^{\mathbf{w}} \quad \text{and} \quad z_2^{\mathbf{w}} = (hy_1)^{\mathbf{w}}(x_3+h)^{\mathbf{w}}. \tag{11.10}$$

Thus, we have $\deg_{\mathbf{w}} z_1 > \deg_{\mathbf{w}} y_1$. Since $\phi(y_2) = y_2$, it follows that

$$\deg_{\mathbf{w}} \phi(x_1) = \deg_{\mathbf{w}} z_1 > \deg_{\mathbf{w}} y_1 > \deg_{\mathbf{w}} y_2 = \deg_{\mathbf{w}} \phi(y_2).$$

Therefore, if $z_i^{\mathbf{w}}$ and $z_j^{\mathbf{w}}$ are algebraically independent over k for some $i, j \in \{1,2,3\}$, then we may conclude that ϕ is wild, because y_2 is a W-test polynomial.

Assume that h belongs to $k^{\times}$. Then, we have $(x_3+h)^{\mathbf{w}} = x_3$. Hence, we get $z_1^{\mathbf{w}} \approx (y_1^{\mathbf{w}})^2x_3$ and $z_2^{\mathbf{w}} \approx y_1^{\mathbf{w}}x_3$ by (11.10). Since $y_1^{\mathbf{w}}$ and x_3 are algebraically independent over k by (11.9), it follows that $z_1^{\mathbf{w}}$ and $z_2^{\mathbf{w}}$ are algebraically independent over k. Therefore, we conclude that ϕ is wild.

Assume that h does not belong to $k^{\times}$. Then, $h^{\mathbf{w}}$ does not belong to $k^{\times}$. Since $\max I = 4 > 3$, we have $\ker D_3 = k[f_3, f_4]$ by Theorem 1.1 (i). By (11.3) and (11.4), we see that $k[f_3, f_4] = k[y_1, y_2]$. Because $y_1^{\mathbf{w}}$ and $y_2^{\mathbf{w}}$ are algebraically independent over k by (11.9), we have $k[y_1, y_2]^{\mathbf{w}} = k[y_1^{\mathbf{w}}, y_2^{\mathbf{w}}]$. Thus, $h^{\mathbf{w}}$ belongs to $k[y_1^{\mathbf{w}}, y_2^{\mathbf{w}}] \setminus k$. This implies that $\deg_{\mathbf{w}} h \geq \deg_{\mathbf{w}} y_i$ for some $i \in \{1,2\}$. Since $\deg_{\mathbf{w}} y_1 > \deg_{\mathbf{w}} y_2 > \deg_{\mathbf{w}} x_3$, it follows that $(x_3+h)^{\mathbf{w}} = h^{\mathbf{w}}$. Therefore, we get $z_1^{\mathbf{w}} = (h^3y_1^2)^{\mathbf{w}}$ and $z_2^{\mathbf{w}} = (h^2y_1)^{\mathbf{w}}$ from (11.10).

Assume that $h^{\mathbf{w}}$ belongs to $k[y_1^{\mathbf{w}}, y_2^{\mathbf{w}}] \setminus k[y_1^{\mathbf{w}}]$. Then, $h^{\mathbf{w}}$ and $y_1^{\mathbf{w}}$ are algebraically independent over k, since so are $y_1^{\mathbf{w}}$ and $y_2^{\mathbf{w}}$. This implies that $z_2^{\mathbf{w}}$ and $z_3^{\mathbf{w}}$ are algebraically independent over k. Therefore, we conclude that ϕ is wild.

In the rest of this section, we consider the case where $h^{\mathbf{w}}$ belongs to $k[y_1^{\mathbf{w}}] \setminus k$. In this case, there exists $l \in \mathbf{N}$ such that $h^{\mathbf{w}} \approx (y_1^{\mathbf{w}})^l$. Then, we have

$$z_1^{\mathbf{w}} \approx (y_1^{\mathbf{w}})^{3l+2}, \quad z_2^{\mathbf{w}} \approx (y_1^{\mathbf{w}})^{2l+1}, \quad z_3^{\mathbf{w}} \approx (y_1^{\mathbf{w}})^{l}, \quad \phi(f_2)^{\mathbf{w}} \approx (y_1^{\mathbf{w}})^{l+1}. \tag{11.11}$$

Put $S = \{y_1, y_2\}$. Then, we have $\deg_{\mathbf{w}}^S h = \deg_{\mathbf{w}} h = l \deg_{\mathbf{w}} y_1$, since $y_1^{\mathbf{w}}$ and $y_2^{\mathbf{w}}$ are algebraically independent over k.

First, assume that $l = 1$. Then, we have $\deg_{\mathbf{w}}^S h = \deg_{\mathbf{w}} y_1 < \deg_{\mathbf{w}} y_2^2$. Hence, we may write

$$h = a_{1,0}y_1 + a_{0,1}y_2 + a_{0,0},$$

where $a_{1,0}, a_{0,1}, a_{0,0} \in k$ with $a_{1,0} \neq 0$. We define $\phi' \in \operatorname{Aut}(k[\mathbf{x}]/k)$ by

$$\phi'(x_1) = a_{1,0}^2z_1 + (z_3^3 - 2a_{1,0}z_2)z_3^2, \quad \phi'(x_2) = a_{1,0}z_2 - z_3^3, \quad \phi'(x_3) = z_3.$$

Then, $\phi^{-1}\circ\phi'$ belongs to $J(k[x_3];x_2,x_1)$. Hence, ϕ is wild if and only if ϕ' is wild. Consider the polynomial

$$\begin{aligned} g := z_3^2 - a_{1,0}\phi(f_2) &= (x_3+h)^2 - a_{1,0}(f_2+hy_1) \\ &= h(2x_3+h-a_{1,0}y_1) - a_{1,0}f_2 + x_3^2 \\ &= h(2x_3+a_{0,1}y_2+a_{0,0}) - a_{1,0}f_2 + x_3^2. \end{aligned}$$

Since $\deg_{\mathbf{w}} h = \deg_{\mathbf{w}} y_1$ is greater than $\deg_{\mathbf{w}} f_2$ and $\deg_{\mathbf{w}} x_3^2$, we see that $g^{\mathbf{w}} = (a_{0,1}hy_2)^{\mathbf{w}} \approx (y_1y_2)^{\mathbf{w}}$ if $a_{0,1}\neq 0$, and $g^{\mathbf{w}} = (2hx_3)^{\mathbf{w}} \approx (y_1x_3)^{\mathbf{w}}$ otherwise. In either case, we have $\deg_{\mathbf{w}} g > \deg_{\mathbf{w}} y_1$, and $g^{\mathbf{w}}$ and $z_3^{\mathbf{w}} \approx y_1^{\mathbf{w}}$ are algebraically independent over k. A direct computation shows that

$$\begin{aligned} \phi'(x_1) &= a_{1,0}^2 z_1 + \bigl(z_3^3 - 2a_{1,0}\bigl(y_2+\phi(f_2)z_3\bigr)\bigr)z_3^2 \\ &= a_{1,0}^2 z_1 + \bigl(z_3^4 - 2a_{1,0}\phi(f_2)z_3^2 - 2a_{1,0}y_2z_3\bigr)z_3 \\ &= a_{1,0}^2 z_1 + \bigl(g^2 - a_{1,0}^2\phi(f_2)^2 - 2a_{1,0}y_2z_3\bigr)z_3 \\ &= (g^2 - 2a_{1,0}y_2z_3)z_3 + a_{1,0}^2\bigl(z_1 - \phi(f_2)^2z_3\bigr) \\ &= (g^2 - 2a_{1,0}y_2z_3)z_3 + a_{1,0}^2\bigl(y_1 + (2y_2+\alpha_2^0)(f_2+hy_1)\bigr), \end{aligned}$$

where we use (11.7) and (11.8) for the first and last equalities. Since $\deg_{\mathbf{w}} g^2z_3 > 3\deg_{\mathbf{w}} y_1 > \deg_{\mathbf{w}} hy_1y_2$, this gives that $\phi'(x_1)^{\mathbf{w}} = (g^2z_3)^{\mathbf{w}}$. By (11.7), we have

$$\phi'(x_2) = a_{1,0}(y_2+\phi(f_2)z_3) - z_3^3 = a_{1,0}y_2 - (z_3^2 - a_{1,0}\phi(f_2))z_3 = a_{1,0}y_2 - gz_3.$$

Hence, we get $\phi'(x_2)^{\mathbf{w}} = (-gz_3)^{\mathbf{w}}$. Thus, $\phi'(x_1)^{\mathbf{w}}$, $\phi'(x_2)^{\mathbf{w}}$ and $\phi'(x_3)^{\mathbf{w}} = z_3^{\mathbf{w}}$ are algebraically dependent over k, and are pairwise algebraically independent over k, since $g^{\mathbf{w}}$ and $z_3^{\mathbf{w}}$ are algebraically independent over k. It is easy to check that $(g^2z_3)^{\mathbf{w}}$, $(gz_3)^{\mathbf{w}}$ and $z_3^{\mathbf{w}}$ do not belong to $k[(gz_3)^{\mathbf{w}}, z_3^{\mathbf{w}}]$, $k[(g^2z_3)^{\mathbf{w}}, z_3^{\mathbf{w}}]$ and $k[(g^2z_3)^{\mathbf{w}}, (gz_3)^{\mathbf{w}}]$, respectively. Therefore, ϕ' satisfies the conditions (1) and (2) after Theorem 1.3. This proves that ϕ' is wild, thereby proving that ϕ is wild.

For the case of $l\geq 2$, we need the following lemma.

Lemma 11.1. *We have* $\deg_{\mathbf{w}} dz_2\wedge dz_3 \geq \deg_{\mathbf{w}} z_2x_3$.

PROOF. Since $z_2 = y_2+\phi(f_2)z_3$, we may write $dz_2\wedge dz_3 = dy_2\wedge dz_3 + z_3\eta$, where $\eta := d\phi(f_2)\wedge dz_3$. We show that $\deg_{\mathbf{w}}\eta \geq \deg_{\mathbf{w}} hy_1x_3$ below. Then, it follows that

$$\deg_{\mathbf{w}} z_3\eta \geq \deg_{\mathbf{w}}\deg_{\mathbf{w}} hy_1x_3z_3 > \deg_{\mathbf{w}} y_2z_3 \geq \deg_{\mathbf{w}} dy_2\wedge dz_3.$$

This implies that $\deg_{\mathbf{w}} dz_2\wedge dz_3 = \deg_{\mathbf{w}} z_3\eta$, and is at least

$$\deg_{\mathbf{w}} hy_1x_3z_3 = (2l+1)\deg_{\mathbf{w}} y_1 + \deg_{\mathbf{w}} x_3 = \deg_{\mathbf{w}} z_2x_3$$

by (11.11). Thus, the lemma is proved.

Now, we show that $\deg_{\mathbf{w}}\eta \geq \deg_{\mathbf{w}} hy_1x_3$. Since $\phi(f_2) = f_2+hy_1$ and $z_3 = x_3+h$, we have

$$\eta = d(f_2+hy_1)\wedge dz_3 = df_2\wedge dz_3 + d(hy_1)\wedge d(x_3+h) = \eta' + hdy_1\wedge dh,$$

where $\eta' := df_2\wedge dz_3 + d(hy_1)\wedge dx_3$. Since $(hy_1)^{\mathbf{w}} \approx (y_1^{\mathbf{w}})^{l+1}$ and x_3 are algebraically independent over k, we have

$$\deg_{\mathbf{w}} d(hy_1)\wedge dx_3 = \deg_{\mathbf{w}} hy_1x_3 > \deg_{\mathbf{w}} f_2z_3 \geq \deg_{\mathbf{w}} df_2\wedge dz_3.$$

Hence, we get $\deg_{\mathbf{w}} \eta' = \deg_{\mathbf{w}} hy_1x_3$. If h belongs to $k[y_1]$, then we have $dy_1 \wedge dh = 0$, and so $\deg_{\mathbf{w}} \eta = \deg_{\mathbf{w}} \eta' = \deg_{\mathbf{w}} hy_1x_3$. Thus, the assertion is true. Assume that h does not belong to $k[y_1]$. Since h is an element of $\ker D_3 = k[y_1, y_2]$, we get

$$dy_1 \wedge dh = \frac{\partial h}{\partial y_2} dy_1 \wedge dy_2 \quad \text{and} \quad \frac{\partial h}{\partial y_2} \neq 0.$$

Since $y_1^{\mathbf{w}}$ and $y_2^{\mathbf{w}}$ are algebraically independent over k, we have $\deg_{\mathbf{w}} dy_1 \wedge dy_2 = \deg_{\mathbf{w}} y_1y_2$. Hence, it follows that

$$\deg_{\mathbf{w}} hdy_1 \wedge dh = \deg_{\mathbf{w}} h\frac{\partial h}{\partial y_2} dy_1 \wedge dy_2 \geq \deg_{\mathbf{w}} hy_1y_2 > \deg_{\mathbf{w}} hy_1x_3 = \deg_{\mathbf{w}} \eta'.$$

Since $\eta = \eta' + hdy_1 \wedge dh$, this implies that $\deg_{\mathbf{w}} \eta = \deg_{\mathbf{w}} hdy_1 \wedge dh$, and is greater than $\deg_{\mathbf{w}} hy_1x_3$. Therefore, the assertion is true. □

Now, assume that $l = 2$. Then, we have $\deg_{\mathbf{w}}^S h = \deg_{\mathbf{w}} h = \deg_{\mathbf{w}} y_1^2$. Since $\deg_{\mathbf{w}} y_2^3 < \deg_{\mathbf{w}} y_1^2 < \deg_{\mathbf{w}} y_2^4$ by (11.9), we may write

$$h = a_{2,0}y_1^2 + a_{0,3}y_2^3 + a_{1,1}y_1y_2 + a_{0,2}y_2^2 + a_{1,0}y_1 + a_{0,1}y_2 + a_{0,0},$$

where $a_{i,j}$'s are elements of k such that $a_{2,0} \neq 0$. Define $\psi \in \mathrm{Aut}(k[\mathbf{x}]/k)$ by

$$\psi(x_1) = f := a_{2,0}z_1 + (a_{1,0}z_2 - z_3^3)z_3$$

and $\psi(x_i) = z_i$ for $i = 2, 3$. Then, $\phi^{-1} \circ \psi$ is elementary. Hence, ϕ is wild if and only if ψ is wild. So we prove that ψ is wild.

In the notation above, we have the following lemma.

Lemma 11.2. (i) $\deg_{\mathbf{w}} h^3x_3 \leq \deg_{\mathbf{w}} f < \deg_{\mathbf{w}} h^4$.
(ii) $f^{\mathbf{w}}$ *and* $y_1^{\mathbf{w}}$ *are algebraically independent over* k.

PROOF. Set $\bar{h} = h - (a_{2,0}y_1^2 + a_{1,0}y_1)$. Then, by (11.7), we have

$$\begin{aligned} g_1 &:= a_{1,0}z_2 - h^2(h - a_{2,0}y_1^2) \\ &= a_{1,0}\big(y_2 + (f_2 + hy_1)(x_3 + h)\big) - h^2(\bar{h} + a_{1,0}y_1) \\ &= a_{1,0}\big(y_2 + f_2(x_3 + h) + hy_1x_3\big) - h^2\bar{h}. \end{aligned}$$

If $\bar{h} \neq a_{0,0}$, then $\bar{h}^{\mathbf{w}}$ is equal to one of $(y_2^3)^{\mathbf{w}}$, $(y_1y_2)^{\mathbf{w}}$, $(y_2^2)^{\mathbf{w}}$ and $y_2^{\mathbf{w}}$ up to nonzero constant multiples, since $y_1^{\mathbf{w}}$ and $y_2^{\mathbf{w}}$ are algebraically independent over k. If this is the case, then we have $\deg_{\mathbf{w}} \bar{h} > \deg_{\mathbf{w}} x_3$. Hence, we get $\deg_{\mathbf{w}} h^2\bar{h} > \deg_{\mathbf{w}} hy_1x_3$. Thus, we know that $g_1^{\mathbf{w}} \approx (h^2\bar{h})^{\mathbf{w}}$ and $\deg_{\mathbf{w}} g_1 > \deg_{\mathbf{w}} h^2x_3$. If $\bar{h} = a_{0,0}$, then we have $\deg_{\mathbf{w}} g_1 \leq \deg_{\mathbf{w}} h^2$. Since $z_3 = x_3 + h$, we have

$$\begin{aligned} g_2 &:= a_{2,0}(f_2 + hy_1)^2 + a_{1,0}z_2 - z_3^3 \\ &= a_{2,0}(f_2^2 + 2f_2hy_1) - (x_3^3 + 3hx_3^2 + 3h^2x_3) + g_1. \end{aligned} \tag{11.12}$$

From this, we see that $(g_2 - g_1)^{\mathbf{w}} \approx (h^2x_3)^{\mathbf{w}}$. Therefore, we get

$$g_2^{\mathbf{w}} \approx g_1^{\mathbf{w}} \approx (h^2\bar{h})^{\mathbf{w}} \ \text{ if } \ \bar{h} \neq a_{0,0}, \ \text{ and } \ g_2^{\mathbf{w}} = (g_2 - g_1)^{\mathbf{w}} \approx (h^2x_3)^{\mathbf{w}} \ \text{ otherwise.}$$

In either case, we have $\deg_{\mathbf{w}} h^2x_3 \leq \deg_{\mathbf{w}} g_2 < \deg_{\mathbf{w}} h^3$. If $\bar{h} \neq a_{0,0}$, then $g_2^{\mathbf{w}} \approx (h^2\bar{h})^{\mathbf{w}}$ is equal to one of $(y_1^4y_2^3)^{\mathbf{w}}$, $(y_1^5y_2)^{\mathbf{w}}$, $(y_1^4y_2^2)^{\mathbf{w}}$ and $(y_1^4y_2)^{\mathbf{w}}$ up to nonzero constant multiples. Since $y_1^{\mathbf{w}}$ and $y_2^{\mathbf{w}}$ are algebraically independent over k, we know that $g_2^{\mathbf{w}}$ and $y_1^{\mathbf{w}}$ algebraically independent over k. The same

holds when $\bar{h} = a_{0,0}$ since $y_1^{\mathbf{w}}$ and x_3 are algebraically independent over k. Thus, it suffices to show that $f^{\mathbf{w}} = (g_2h)^{\mathbf{w}}$. By (11.8), we have

$$f = a_{2,0}z_1 + g_2z_3 - a_{2,0}(f_2 + hy_1)^2z_3 = a_{2,0}\big(y_1 + (2y_2 + \alpha_2^0)(f_2 + hy_1)\big) + g_2z_3.$$

Since $\deg_{\mathbf{w}} h^2x_3 \le \deg_{\mathbf{w}} g_2$, it follows that

$$\deg_{\mathbf{w}}(f - g_2z_3) = \deg_{\mathbf{w}} hy_1y_2 < \deg_{\mathbf{w}} h^3x_3 \le \deg_{\mathbf{w}} g_2h = \deg_{\mathbf{w}} g_2z_3.$$

Therefore, we get $f^{\mathbf{w}} = (g_2z_3)^{\mathbf{w}} = (g_2h)^{\mathbf{w}}$. $\square$

We prove that ψ admits no elementary reduction and no Shestakov-Umirbaev reduction for the weight $\mathbf{w}$. Then, it follows that ψ is wild due to Theorem 1.3. By (11.11) with $l = 2$, we have $z_2^{\mathbf{w}} \approx (y_1^{\mathbf{w}})^5$ and $z_3^{\mathbf{w}} \approx (y_1^{\mathbf{w}})^2$. Hence, $f^{\mathbf{w}}$ and $z_i^{\mathbf{w}}$ are algebraically independent over k for $i = 2, 3$ in view of Lemma 11.2 (ii). Thus, we know that $k[f, z_i]^{\mathbf{w}} = k[f^{\mathbf{w}}, z_i^{\mathbf{w}}]$ for $i = 2, 3$. Since $z_i^{\mathbf{w}}$ does not belong to $k[z_j^{\mathbf{w}}]$ for $(i, j) = (2, 3), (3, 2)$, we see that $z_i^{\mathbf{w}}$ does not belong to $k[f^{\mathbf{w}}, z_j^{\mathbf{w}}]$ for $(i, j) = (2, 3), (3, 2)$. Therefore, $\psi(x_i)^{\mathbf{w}}$ does not belong to $k[\psi(x_1), \psi(x_j)]^{\mathbf{w}}$ for $(i, j) = (2, 3), (3, 2)$. To conclude that ψ admits no elementary reduction for the weight $\mathbf{w}$, it suffices to check that $f^{\mathbf{w}}$ does not belong to $k[z_2, z_3]^{\mathbf{w}}$.

Suppose to the contrary that $f^{\mathbf{w}}$ belongs to $k[z_2, z_3]^{\mathbf{w}}$. Then, there exists $g \in k[z_2, z_3]$ such that $g^{\mathbf{w}} = f^{\mathbf{w}}$. By Lemma 11.2 (ii), $f^{\mathbf{w}}$ cannot be an element of $k[y_1^{\mathbf{w}}]$. Hence, $g^{\mathbf{w}}$ does not belong to $k[z_2^{\mathbf{w}}, z_3^{\mathbf{w}}]$. This implies that $\deg_{\mathbf{w}} g < \deg_{\mathbf{w}}^T g$, where $T := \{z_2, z_3\}$. By (i) and (ii) of Lemma 2.2, there exist $p, q \in \mathbf{N}$ with $\gcd(p, q) = 1$ such that $(z_2^{\mathbf{w}})^p \approx (z_3^{\mathbf{w}})^q$ and

$$\deg_{\mathbf{w}} f = \deg_{\mathbf{w}} g \ge q \deg_{\mathbf{w}} z_3 + \deg_{\mathbf{w}} dz_2 \wedge dz_3 - \deg_{\mathbf{w}} z_2 - \deg_{\mathbf{w}} z_3.$$

Then, we have $(p, q) = (2, 5)$. Since $\deg_{\mathbf{w}} dz_2 \wedge dz_3 \ge \deg_{\mathbf{w}} z_2x_3$ by Lemma 11.1, the right-hand side of the preceding inequality is at least

$$5 \deg_{\mathbf{w}} z_3 + \deg_{\mathbf{w}} z_2x_3 - \deg_{\mathbf{w}} z_2 - \deg_{\mathbf{w}} z_3 > 4 \deg_{\mathbf{w}} z_3 = \deg_{\mathbf{w}} h^4.$$

Hence, we get $\deg_{\mathbf{w}} f > \deg_{\mathbf{w}} h^4$. This contradicts Lemma 11.2 (i). Thus, $f^{\mathbf{w}}$ does not belong to $k[z_2, z_3]^{\mathbf{w}}$. Therefore, ψ admits no elementary reduction for the weight $\mathbf{w}$.

Next, we check that ψ admits no Shestakov-Umirbaev reduction for the weight $\mathbf{w}$ by means of Lemma 1.2. By Lemma 11.2 (i), we have $\deg_{\mathbf{w}} f \ge \deg_{\mathbf{w}} h^3x_3 > 6 \deg_{\mathbf{w}} y_1$. Since $\deg_{\mathbf{w}} z_2 = 5 \deg_{\mathbf{w}} y_1$ and $\deg_{\mathbf{w}} z_3 = 2 \deg_{\mathbf{w}} y_1$, it follows that $\deg_{\mathbf{w}} \psi(x_1) > \deg_{\mathbf{w}} \psi(x_2) > \deg_{\mathbf{w}} \psi(x_3)$. Clearly, $3 \deg_{\mathbf{w}} \psi(x_2) = 15 \deg_{\mathbf{w}} y_1$ is not equal to $4 \deg_{\mathbf{w}} \psi(x_3) = 8 \deg_{\mathbf{w}} y_1$. Since $\operatorname{rank} \mathbf{w} = 3$, we know by Lemma 11.2 (ii) that $\deg_{\mathbf{w}} f$ and $\deg_{\mathbf{w}} y_1$ are linearly independent. Hence, $2 \deg_{\mathbf{w}} \psi(x_1)$ is not equal to $m \deg_{\mathbf{w}} \psi(x_i)$ for any $m \in \mathbf{N}$ and $i = 2, 3$. Thus, we conclude from Lemma 1.2 that ψ admits no Shestakov-Umirbaev reduction for the weight $\mathbf{w}$. This proves that ψ is wild, and thereby proving that ϕ is wild.

Finally, we prove that ϕ is wild when $l \ge 3$ using Lemma 2.2. Define $\iota \in \operatorname{Aut}(k[\mathbf{x}]/k[x_2, x_3])$ by $\iota(x_1) = -x_1$. First, we show that $D' := \iota^{-1} \circ (f_\theta D_\theta) \circ \iota$ is equal to $-f_2\tilde{D}$. Since $\iota^{-1}(f_\theta) = -x_1x_3 + \theta(x_2) = -f_2$ and $D_\theta(-x_1) = \theta'(x_2) = \tilde{D}(x_1)$, we have

$$D'(x_1) = \iota^{-1}(f_\theta)\iota^{-1}\big(D_\theta(-x_1)\big) = -f_2\theta'(x_2) = -f_2\tilde{D}(x_1).$$

Since $D_\theta(x_2) = \tilde{D}(x_2) = x_3$, $D_\theta(x_3) = \tilde{D}(x_3) = 0$ and $\iota(x_i) = x_i$ for $i = 2, 3$, we have $D'(x_i) = \iota^{-1}(f_\theta)\iota^{-1}\big(D_\theta(x_i)\big) = -f_2\tilde{D}(x_i)$ for $i = 2, 3$. Hence, we get $D' = -f_2\tilde{D}$. Thus, it follows that

$$\iota^{-1} \circ \sigma_\theta \circ \iota = \iota^{-1} \circ (\exp f_\theta D_\theta) \circ \iota = \exp D' = \exp(-f_2\tilde{D}) = \tilde{\sigma}.$$

Since $\phi = \exp hD_3$ fixes $f_3 = \tilde{\sigma}(x_1)$, we know that $\phi' := \iota \circ \phi \circ \iota^{-1}$ fixes $\iota(\tilde{\sigma}(x_1)) = \sigma_\theta(\iota(x_1)) = -\sigma_\theta(x_1)$. Therefore, ϕ' belongs to $\mathrm{Aut}(k[\mathbf{x}]/k[\sigma_\theta(x_1)])$. So we prove that ϕ' is wild by means of Lemma 2.2. Then, it follows that ϕ is wild.

Define $\gamma_0^{\mathbf{w}}, \ldots, \gamma_3^{\mathbf{w}}$ as in Section 2 for ϕ'. Thanks to Lemma 2.2, it suffices to verify that

$$\gamma_0^{\mathbf{w}} < \gamma_1^{\mathbf{w}}, \quad \gamma_2^{\mathbf{w}} > \gamma_3^{\mathbf{w}}, \quad 2\gamma_1^{\mathbf{w}} > 3\gamma_2^{\mathbf{w}} + 2\gamma_3^{\mathbf{w}} \quad \text{and} \quad \gamma_1^{\mathbf{w}} \geq \gamma_2^{\mathbf{w}} + 2\gamma_3^{\mathbf{w}}. \tag{11.13}$$

For $i = 2, 3$, we have $\phi'(x_i) = \iota(\phi(x_i)) = \iota(z_i)$. Since ι preserves the $\mathbf{w}$-degree of each element of $k[\mathbf{x}]$, it follows that $\deg_{\mathbf{w}} \phi'(x_i) = \deg_{\mathbf{w}} z_i$ for $i = 2, 3$. Since $\phi'(f_\theta) = \iota(\phi(-f_2)) = -\iota(\phi(f_2))$, we have $\deg_{\mathbf{w}} \phi'(f_\theta) = \deg_{\mathbf{w}} \phi(f_2)$ similarly. Put $\gamma = \deg_{\mathbf{w}} y_1$. Then, it follows from (11.11) that

$$\deg_{\mathbf{w}} \phi'(x_2) = (2l+1)\gamma, \ \deg_{\mathbf{w}} \phi'(x_3) = l\gamma, \ \deg_{\mathbf{w}} \phi'(f_\theta) = (l+1)\gamma. \tag{11.14}$$

Since $l \geq 3$ by assumption, $2l + 1$ is not a multiple of l. Hence, we know that $\phi'(x_2)^{\mathbf{w}}$ does not belong to $k[\phi'(x_3)]^{\mathbf{w}}$. By the definition of $\gamma_2^{\mathbf{w}}$, this implies that $\gamma_2^{\mathbf{w}} = \deg_{\mathbf{w}} \phi'(x_2)$. Hence, $\gamma_2^{\mathbf{w}}$ is greater than $\gamma_3^{\mathbf{w}} = \deg_{\mathbf{w}} \phi'(x_3)$ by (11.14). This proves the second part of (11.13). Since $l + 1$ is not a multiple of l, we know that $\phi'(f_\theta)^{\mathbf{w}}$ does not belong to $k[\phi'(x_3)]^{\mathbf{w}}$. By the definition of $\gamma_0^{\mathbf{w}}$, this implies that $\gamma_0^{\mathbf{w}} = \deg_{\mathbf{w}} \phi'(f_\theta)$. Hence, $\gamma_0^{\mathbf{w}}$ is less than $\gamma_2^{\mathbf{w}} = \deg_{\mathbf{w}} \phi'(x_2)$ by (11.14). By the definition of $\gamma_1^{\mathbf{w}}$, we may write $\gamma_1^{\mathbf{w}} = \eta(\theta; \phi'(x_3), \phi'(x_2))$ in the notation of Lemma 2.3. Note that

$$\theta(\phi'(x_2)) + \rho\big(\phi'(x_2), \phi'(x_3)\big)\phi'(x_3) = \iota\big(\theta(z_2) + \rho(z_2, z_3)z_3\big)$$

for any $\rho(y, z) \in k[y, z]$. Since ι preserves the $\mathbf{w}$-degree, it follows that

$$\gamma_1^{\mathbf{w}} = \eta(\theta; \phi'(x_3), \phi'(x_2)) = \eta(\theta; z_3, z_2).$$

By (11.11), we see that (1) and (2) before Lemma 2.3 are fulfilled for $f = z_3$ and $g = z_2$. Clearly, (3) is satisfied. Moreover, the condition (i) of Lemma 2.3 is satisfied, because $\deg_z \theta(z) = 2$, $\deg_{\mathbf{w}} dz_3 \wedge dz_2 > \deg_{\mathbf{w}} z_2$ by Lemma 11.1, $(2l + 1)\deg_{\mathbf{w}} z_3 = l \deg_{\mathbf{w}} z_2$ by (11.11), and $l \geq 3$ by assumption. Thus, we conclude that $\gamma_1^{\mathbf{w}} = \eta(\theta; z_3, z_2)$ is greater than

$$\deg_{\mathbf{w}} z_3 + \frac{3}{2}\deg_{\mathbf{w}} z_2 \quad \text{and} \quad 2\deg_{\mathbf{w}} z_3 + \deg_{\mathbf{w}} z_2.$$

Since $\deg_{\mathbf{w}} z_i = \deg_{\mathbf{w}} \phi'(x_i) = \gamma_i^{\mathbf{w}}$ for $i = 2, 3$, this implies the last two parts of (11.13). Since $\gamma_0^{\mathbf{w}} < \gamma_2^{\mathbf{w}}$ as mentioned, it follows that $\gamma_0^{\mathbf{w}} < \gamma_1^{\mathbf{w}}$. Therefore, we get the four inequalities of (11.13). This proves that ϕ' is wild, and thereby proving that ϕ is wild.

This completes the proof of Theorem 1.1 (iii) in the case of (w2), and thus completing the proof of Theorem 1.1.

12. Plinth ideals (I)

The goal of this section is to prove Theorem 1.3. First, we remark that

$$\operatorname{pl} \sigma \circ D \circ \sigma^{-1} = \sigma(\operatorname{pl} D)$$

for each $\sigma \in \operatorname{Aut}(k[\mathbf{x}]/k)$ and $D \in \operatorname{Der}_k k[\mathbf{x}]$, since

$$\ker \sigma \circ D \circ \sigma^{-1} = \sigma(\ker D) \quad \text{and} \quad (\sigma \circ D \circ \sigma^{-1})(k[\mathbf{x}]) = \sigma(D(k[\mathbf{x}])).$$

By definition, we have

$$\operatorname{rank} \sigma \circ D \circ \sigma^{-1} = \operatorname{rank} D.$$

Lemma 12.1. *For $D \in \operatorname{LND}_k k[\mathbf{x}]$, the following conditions are equivalent*:
(1) $\operatorname{pl} D = \ker D$.
(2) $D(s) = 1$ *for some* $s \in k[\mathbf{x}]$.
(3) D *is irreducible and* $\operatorname{rank} D = 1$.

Proof. By the definition of $\operatorname{pl} D$, we see that (1) and (2) are equivalent. Since $D(s) = 1$ belongs to no proper ideal of $k[\mathbf{x}]$, (2) implies that D is irreducible. Recall that, if $D(s) = 1$ for $D \in \operatorname{LND}_k k[\mathbf{x}]$ and $s \in k[\mathbf{x}]$, then we have $k[\mathbf{x}] = (\ker D)[s]$ (cf. [**6**, Proposition 1.3.21]). By Miyanishi [**21**], we have $\ker D = k[f, g]$ for some $f, g \in k[\mathbf{x}]$ for any $D \in \operatorname{LND}_k k[\mathbf{x}] \setminus \{0\}$. Hence, (2) implies that $k[f, g, s] = k[\mathbf{x}]$ for some $f, g \in k[\mathbf{x}]$, and so implies that $\operatorname{rank} D = 1$. Thus, (2) implies (3). If $\operatorname{rank} D = 1$ for $D \in \operatorname{LND}_k k[\mathbf{x}]$, then we have $\sigma \circ D \circ \sigma^{-1} = f\partial/\partial x_1$ for some $\sigma \in \operatorname{Aut}(k[\mathbf{x}]/k)$ and $f \in k[x_2, x_3] \setminus \{0\}$. If furthermore D is irreducible, then f belongs to $k^{\times}$. When this is the case, $s := f^{-1}\sigma(x_1)$ belongs to $k[\mathbf{x}]$, and satisfies $D(s) = 1$. Thus, (3) implies (2). Therefore, (1), (2) and (3) are equivalent. □

We begin with the proof of (i) and (ii) of Theorem 1.3. If $t_0 = 1$, then we have $I = \{1\}$. From (1.4), we see that $\operatorname{rank} D_1 = 1$ and

$$\operatorname{pl} D_1 = x_1 k[\mathbf{x}] \cap k[x_1, x_3] = x_1 k[x_1, x_3] = x_1 \ker D_1,$$

proving the first part of (i). Hence, we get $\operatorname{rank} D_1' = 1$ and $\operatorname{pl} D_1' = x_1 \ker D_1'$ if $t_1 = 1$. If furthermore $t_0 = 2$, then we have $D_2 = \tau_2 \circ D_1' \circ \tau_2^{-1}$ by Theorem 1.1 (ii). Thus, it follows that $\operatorname{rank} D_2 = 1$ and

$$\operatorname{pl} D_2 = \tau_2(\operatorname{pl} D_1') = \tau_2(x_1 \ker D_1') = f_2 \ker D_2,$$

proving the second part of (i). If $(t_0, i) = (2, 1)$ or $i = 0$, then we have $D_i(x_3) = 1$. Hence, we get $\operatorname{pl} D_i = \ker D_i$ and $\operatorname{rank} D_i = 1$ by Lemma 12.1. By the last part of Theorem 1.1 (ii), the same holds for every $i \geq 1$ if $t_0 = t_1 = 2$, proving (ii).

To analyze the detailed structure of plinth ideals, the following lemma is useful. For an irreducible element p of $k[\mathbf{x}]$, we consider the *p-adic valuation* v_p of $k(\mathbf{x})$, i.e., the map $v_p : k(\mathbf{x}) \to \mathbf{Z} \cup \{\infty\}$ defined by $v_p(f) = m$ for each $f \in k(\mathbf{x}) \setminus \{0\}$ and $v_p(0) = \infty$, where $m \in \mathbf{Z}$ is such that $f = p^m g/h$ for some $g, h \in k[\mathbf{x}] \setminus pk[\mathbf{x}]$. Now, take $D \in \operatorname{LND}_k k[\mathbf{x}]$ and $s \in k[\mathbf{x}]$ such that $D(s) \neq 0$ and $D^2(s) = 0$. Then, $D(s)$ belongs to $\operatorname{pl} D$. Since $\operatorname{pl} D$ is a nonzero principal ideal of $\ker D$, and $D(s) \neq 0$ by assumption, we may find a factor q of $D(s)$ (may be an element of $k^{\times}$) such that $\operatorname{pl} D = q \ker D$. Let $p \in k[\mathbf{x}]$ be any factor of $D(s)$ which is an irreducible element of $k[\mathbf{x}]$. Then, p belongs to $\ker D$ by the factorially closedness of $\ker D$ in $k[\mathbf{x}]$. Note that

$i := v_p(q)$ is the maximal number such that $p^{-i}q$ belongs to $k[\mathbf{x}]$. Since $p^{-i}q$ belongs to $\ker D$, we see that i is equal to the maximal number such that q belongs to $p^i \ker D$. Thus, we know that

$$i = v_p(q) = \max\{i \geq 0 \mid \mathrm{pl}\, D \text{ is contained in } p^i \ker D\}.$$

We define

$$j = \max\{j \geq 0 \mid (s + \ker D) \cap p^j k[\mathbf{x}] \neq \emptyset\}.$$

In the notation above, we have the following lemma.

Lemma 12.2. *It holds that* $i + j = v_p(D(s))$. *Hence,* $\mathrm{pl}\, D$ *is contained in* $p \ker D$ *if* $(s + \ker D) \cap pk[\mathbf{x}] = \emptyset$.

Proof. Put $l = v_p(D(s)) - v_p(q)$. We show that

$$S_1 := (s + \ker D) \cap p^l k[\mathbf{x}] \neq \emptyset \quad \text{and} \quad S_2 := (s + \ker D) \cap p^{l+1} k[\mathbf{x}] = \emptyset.$$

Then, it follows that $l = j$. Hence, we get $i + j = v_p(q) + l = v_p(D(s))$. Since $q \ker D = \mathrm{pl}\, D$, we have $q = D(t)$ for some $t \in k[\mathbf{x}]$. Since $D(s)$ belongs to $\mathrm{pl}\, D$, there exists $h_1 \in \ker D$ such that $D(s) = qh_1 = D(t)h_1$. Then, it follows that $D(s - th_1) = 0$. Hence, $h_2 := s - th_1$ belongs to $\ker D$. Thus, $th_1 = s - h_2$ belongs to $s + \ker D$. Since $h_1 = D(s)q^{-1}$, we have

$$v_p(th_1) = v_p\big(tD(s)q^{-1}\big) \geq v_p(D(s)) - v_p(q) = l.$$

Hence, th_1 belongs to $p^l k[\mathbf{x}]$. Thus, th_1 belongs to S_1. Therefore, S_1 is not empty. Next, suppose to the contrary that S_2 is not empty. Then, we have $s + h = p^{l+1}u$ for some $h \in \ker D$ and $u \in k[\mathbf{x}]$. Since $D(h) = D(p) = 0$, it follows that $D(u) = D(s)p^{-l-1}$. Hence, we get

$$v_p(D(u)) = v_p(D(s)p^{-l-1}) = v_p(D(s)) - l - 1 = v_p(q) - 1.$$

On the other hand, $D(u)$ belongs to $\mathrm{pl}\, D$, since $D^2(u) = D^2(s)p^{-l-1} = 0$. Hence, we have $D(u) = qh'$ for some $h' \in \ker D$. Thus, we get $v_p(D(u)) \geq v_p(q)$, a contradiction. Therefore, S_2 is empty.

To prove the last part, assume that $(s + \ker D) \cap pk[\mathbf{x}] = \emptyset$. Then, we have $j = 0$. Since $v_p(D(s)) = i + j$, it follows that $i = v_p(D(s))$. Because p is a factor of $D(s)$, we have $v_p(D(s)) \geq 1$. Thus, we get $i \geq 1$. Therefore, $\mathrm{pl}\, D$ is contained in $p \ker D$. $\square$

We remark that the former case of Theorem 1.3 (iii) is reduced to the latter case thanks to Theorem 1.1 (ii). In fact, since $t_0 = 2$, we have $D_2 = \tau_2 \circ D_1' \circ \tau_2^{-1}$ by Theorem 1.1 (ii). Since $t_1 \geq 3$, it follows that $\mathrm{rank}\, D_2 = \mathrm{rank}\, D_1' = 2$ by the latter case of Theorem 1.3 (iii). Since $\tau_2(f_1') = f_2$ by Lemma 2.1 (i), the latter case of Theorem 1.3 (iii) implies that

$$\mathrm{pl}\, D_2 = \tau_2(\mathrm{pl}\, D_1') = \tau_2(f_1' \ker D_1') = \tau_2(f_1')\tau_2(\ker D_1') = f_2 \ker D_2.$$

Similarly, the former case of Theorem 1.3 (iv) is reduced to the latter case.

Now, we prove Theorem 1.3 (iii). By the remark, we may assume that $t_0 \geq 3$ and $i = 1$. Note that $D_1(x_2) = x_1$ and $D_1^2(x_2) = 0$. Hence, we have $\mathrm{pl}\, D_1 = (q)$ for some factor q of x_1 by the discussion before Lemma 12.2. We prove that $q \approx x_1$. Then, it follows that $\mathrm{pl}\, D_1 = (x_1) = (f_1)$. Since D_1 is irreducible when $t_0 \geq 3$ by (b) of Theorem 1.1 (i), this implies that $\mathrm{rank}\, D_1 \geq 2$ because of Lemma 12.1. Since $D_1(x_1) = 0$, we have $\mathrm{rank}\, D_1 \leq 2$. Hence, we get $\mathrm{rank}\, D_1 = 2$. Thus, the proof is completed. Now, suppose

to the contrary that $q \not\approx x_1$. Then, q must be an element of $k^\times$. Hence, we have $\operatorname{pl} D_1 = \ker D_1$, and is not contained in $x_1 \ker D_1$. Since x_1 is an irreducible factor of $D_1(x_2)$, it follows that $(x_2 + \ker D_1) \cap x_1 k[\mathbf{x}] \neq \emptyset$ by the last part of Lemma 12.2. Take $g_1 \in \ker D_1$ and $g_2 \in k[\mathbf{x}]$ such that $x_2 + g_1 = x_1 g_2$. Since $\ker D_1 = k[x_1, f_2]$ by Theorem 1.1 (i), we may write $g_1 = \nu(x_1, f_2)$, where $\nu(y,z) \in k[y,z]$. Then, we have $x_2 = -\nu(x_1, f_2) + x_1 g_2$. Substituting zero for x_1, we get $x_2 = -\nu\bigl(0, -\theta(x_2)\bigr)$. Hence, $\nu(0,z)$ does not belong to k. Thus, we know that

$$1 = \deg_{x_2} x_2 = (\deg_z \nu(0,z)) \deg_{x_2} \theta(x_2) \geq \deg_{x_2} \theta(x_2) = t_0 - 1 \geq 2,$$

a contradiction. This proves that $q \approx x_1$, and thereby completing the proof of (iii).

To prove (iv), we need the following result.

Lemma 12.3. *Assume that $D \in \mathrm{LND}_k\, k[\mathbf{x}] \setminus \{0\}$ is irreducible. If $D(p) = 0$ for a coordinate p of $k[\mathbf{x}]$ over k, then we have $D(k[\mathbf{x}]) \cap k[p] \neq \{0\}$. Consequently, we have $\operatorname{pl} D = (q)$ for some $q \in k[p] \setminus \{0\}$.*

Proof. Without loss of generality, we may assume that $p = x_1$. Then, D extends to a locally nilpotent derivation $\tilde{D}$ of $R := k(x_1)[x_2, x_3]$ over $k(x_1)$. We show that $\tilde{D}$ is irreducible by contradiction. Suppose to the contrary that $\tilde{D}$ is not irreducible. Then, we may find $q \in R \setminus k(x_1)$ such that $\tilde{D}(R)$ is contained in qR. Multiplying by an element of $k(x_1)^\times$, we may assume that q belongs to $k[\mathbf{x}] \setminus k[x_1]$, and is not divisible by any element of $k[x_1] \setminus k$. Then, qh does not belong to $k[\mathbf{x}]$ for any $h \in R \setminus k[\mathbf{x}]$. Hence, $qR \cap k[\mathbf{x}]$ is contained in $qk[\mathbf{x}]$. Since $D(k[\mathbf{x}]) = \tilde{D}(k[\mathbf{x}])$ is contained in $qR \cap k[\mathbf{x}]$, it follows that $D(k[\mathbf{x}])$ is contained in $qk[\mathbf{x}]$. This contradicts that D is irreducible. Therefore, $\tilde{D}$ is irreducible.

Now, by Theorem 2, there exist $\tau \in \mathrm{Aut}(R/k(x_1))$ and $f \in k(x_1)[x_2] \setminus \{0\}$ such that $D' := \tau^{-1} \circ \tilde{D} \circ \tau = f\partial/\partial x_3$. Since $\tilde{D}$ is irreducible, so is D'. Hence, f belongs to $k(x_1)^\times$. Take $g \in k(x_1)^\times$ such that fg and $s := \tau(x_3) g$ belong to $k[\mathbf{x}]$. Then, fg belongs to $k[x_1] \setminus \{0\}$. Since $\tau^{-1}(s) = x_3 g$ and $D'(x_3 g) = fg$, it follows that

$$D(s) = \tau\bigl(D'(\tau^{-1}(s))\bigr) = \tau\bigl(D'(x_3 g)\bigr) = \tau(fg) = fg.$$

Thus, $D(s)$ belongs to $k[x_1] \setminus \{0\}$. Therefore, we get $D(k[\mathbf{x}]) \cap k[x_1] \neq \{0\}$.

Since $D \neq 0$ by assumption, we have $\operatorname{pl} D = (q)$ for some $q \in k[\mathbf{x}] \setminus \{0\}$. Take any $h \in D(k[\mathbf{x}]) \cap k[x_1] \setminus \{0\}$. Then, h belongs to $\operatorname{pl} D$. Hence, q is a factor of h. Since $k[x_1]$ is factorially closed in $k[\mathbf{x}]$, it follows that q belongs to $k[x_1] \setminus \{0\}$. $\square$

Using Lemma 12.3, we get the following proposition.

Proposition 12.4. *Let $D \in \mathrm{LND}_k\, k[\mathbf{x}] \setminus \{0\}$ and $p_1, p_2, q \in k[\mathbf{x}]$ be such that $\operatorname{pl} D = (q)$, and p_1 and p_2 are factors of q. If D is irreducible, and p_1 and p_2 are algebraically independent over k, then we have $\operatorname{rank} D = 3$.*

Proof. Suppose to the contrary that $\operatorname{rank} D \leq 2$. Then, we have $D(p) = 0$ for some coordinate p of $k[\mathbf{x}]$ over k. Since D is irreducible by assumption, it follows that $\operatorname{pl} D = (q')$ for some $q' \in k[p] \setminus \{0\}$ by Lemma 12.3. Then, we have $q \approx q'$. Hence, q belongs to $k[p] \setminus \{0\}$. Since $k[p]$ is factorially

closed in $k[\mathbf{x}]$, we know that p_1 and p_2 belong to $k[p]$. Thus, p_1 and p_2 are algebraically dependent over k, a contradiction. Therefore, we conclude that $\operatorname{rank} D = 3$. $\square$

Let $D \in \mathrm{LND}_k\, k[\mathbf{x}] \setminus \{0\}$ and $q \in k[\mathbf{x}]$ be such that D is irreducible and $\operatorname{pl} D = (q)$, and let K be the subfield of $k(\mathbf{x})$ generated over k by all the factors of q. Then, we have $\operatorname{trans.deg}_k K \leq 2$, since every factor of q belongs to $\ker D$, and $\operatorname{trans.deg}_k \ker D = 2$. By Lemma 12.1, we have $\operatorname{trans.deg}_k K = 0$ if and only if $\operatorname{rank} D = 1$. Hence, we know by Proposition 12.4 that $\operatorname{trans.deg}_k K = 1$ if $\operatorname{rank} D = 2$. We do not know whether $\operatorname{rank} D = 2$ whenever $\operatorname{trans.deg}_k K = 1$.

Now, let us prove Theorem 1.3 (iv). Without loss of generality, we may assume that $t_0 \geq 3$, $(t_0, t_1) \neq (3, 1)$ and $i \geq 2$ as remarked. Since $I = \mathbf{N}$, we see that i belongs to I, and is not the maximum of I. Hence, D_i is irreducible, $\ker D_i = k[f_i, f_{i+1}]$ and $D_i(r) = f_i f_{i+1}$ by Theorem 1.1 (i). Since $D_i^2(r) = 0$, we have $\operatorname{pl} D_i = (q)$ for some factor q of $D_i(r) = f_i f_{i+1}$ by the discussion before Lemma 12.2. We show that $q \approx f_i f_{i+1}$. Since f_i and f_{i+1} are irreducible elements of $k[\mathbf{x}]$ with $f_i \not\approx f_{i+1}$ by Lemma 4.1, it suffices to verify that $v_{f_l}(q) = 1$ for $l = i, i+1$. Suppose to the contrary that $v_{f_l}(q) = 0$ for some $l \in \{i, i+1\}$. Then, $\operatorname{pl} D_i = (q)$ is not contained in $f_l \ker D_i$. Hence, $(r + \ker D_i) \cap f_l k[\mathbf{x}]$ is not empty due to the last part of Lemma 12.2. Since $\ker D_i = k[f_i, f_{i+1}]$, it follows that $(r + k[f_j]) \cap f_l k[\mathbf{x}] \neq \emptyset$, where $j \in \{i, i+1\}$ with $j \neq l$. By the last part of Lemma 6.2, this implies that $a_j < 2$. However, since $t_0 \geq 3$, $(t_0, t_1) \neq (3, 1)$ and $j \geq i \geq 2$, we have $a_j \geq 2$ by Lemma 3.1 (ii), a contradiction. This proves that $q \approx f_i f_{i+1}$. Therefore, we get $\operatorname{pl} D_i = (f_i f_{i+1})$. Since f_i and f_{i+1} are algebraically independent over k, it follows that $\operatorname{rank} D_i = 3$ by Proposition 12.4. This completes the proof of (iv).

Finally, we prove Theorem 1.3 (v). Assume that $(t_0, t_1) = (3, 1)$. Then, we can define $\sigma_3 \in \operatorname{Aut}(k[\mathbf{x}]/k)$ as in (11.5). Since $f_2 = \sigma_3(f_2)$, $f_3 = \sigma_3(x_1) = \sigma_3(f_1)$ and $f_4 = \sigma_3(x_2) = \sigma_3(f_0)$, we have

$$
\begin{aligned}
D_2 &= -\Delta_{(f_2, f_3)} = -\Delta_{(\sigma_3(f_2), \sigma_3(f_1))} = -(\det J\sigma_3)(\sigma_3 \circ D_1 \circ \sigma_3^{-1}) \\
D_3 &= -\Delta_{(f_3, f_4)} = -\Delta_{(\sigma_3(f_1), \sigma_3(f_0))} = -(\det J\sigma_3)(\sigma_3 \circ D_0 \circ \sigma_3^{-1})
\end{aligned}
$$

by the formula (2.1). Since $t_0 = 3$, we have $\operatorname{pl} D_1 = f_1 \ker D_1$ and $\operatorname{rank} D_1 = 2$ by Theorem 1.3 (iii). Hence, it follows that

$$
\operatorname{pl} D_2 = \sigma_3(f_1 \ker D_1) = \sigma_3(f_1)\sigma_3(\ker D_1) = f_3 \ker D_2
$$

and $\operatorname{rank} D_2 = 2$. Since $\operatorname{pl} D_0 = \ker D_0$ and $\operatorname{rank} D_0 = 1$, we get $\operatorname{pl} D_3 = \ker D_3$ and $\operatorname{rank} D_3 = 1$ similarly. Recall that $D_4 = f_4 \Delta_{(x_3, f_4)}$ by (11.6). Since $x_3 = \sigma_3(x_3)$ and $f_4 = \sigma_3(x_2)$, we have

$$
\Delta := \Delta_{(x_3, f_4)} = \Delta_{(\sigma_3(x_3), \sigma_3(x_2))} = (\det J\sigma_3)\big(\sigma_3 \circ \Delta_{(x_3, x_2)} \circ \sigma_3^{-1}\big).
$$

Since $\Delta_{(x_3, x_2)} = -\partial/\partial x_1$, it follows that $\operatorname{pl} \Delta = \ker \Delta$ and $\operatorname{rank} \Delta = 1$. Therefore, we conclude that $\operatorname{pl} D_4 = f_4 \ker D_4$ and $\operatorname{rank} D_4 = 1$. This completes the proof of (v), and thus completing the proof of Theorem 1.3.

13. Plinth ideals (II)

In this section, we prove Theorem 1.6. Assume that $t_0 \geq 3$ and $i = 2$, or $t_0 \geq 3$, $(t_0, t_1) \neq (3, 1)$ and $i \geq 3$. By Theorem 1.5 (i), $\tilde{D}_i$ is irreducible and locally nilpotent, and satisfies $\ker \tilde{D}_i = k[f_i, \tilde{f}_{i+1}]$. Hence, f_i and $\tilde{f}_{i+1}$ are irreducible elements of $k[\mathbf{x}]$ by Lemma 4.1. From (1.9), we see that $\tilde{f}_{i+1}$ is a factor of $\tilde{D}_i(r_i)$. If $i \geq 3$, then f_i is also a factor of $\tilde{D}_i(r_i)$. Since $\tilde{D}_i^2(r_i) = 0$, we have $\operatorname{pl} \tilde{D}_i = (\tilde{g}_i)$ for some factor $\tilde{g}_i$ of $\tilde{D}_i(r_i)$ by the discussion before Lemma 12.2.

We note that $a_i \geq 2$ under the assumption above. Actually, we have $a_2 = t_0 - 1 \geq 2$ if $i = 2$, and $a_i \geq 2$ if $i \geq 3$ by Lemma 3.1 (ii).

Lemma 13.1. *$\tilde{f}_{i+1}$ is a factor of $\tilde{g}_i$. Hence, we have* $\operatorname{rank} \tilde{D}_i \geq 2$. *If $\tilde{g}_i$ and $\tilde{D}_i(r_i)\tilde{f}_{i+1}^{-1}$ have a non-constant common factor, then we have* $\operatorname{rank} \tilde{D}_i = 3$.

Proof. Suppose to the contrary that $\tilde{f}_{i+1}$ is not a factor of $\tilde{g}_i$. Then, $\operatorname{pl} \tilde{D}_i$ is not contained in $\tilde{f}_{i+1} \ker \tilde{D}_i$. By the last part of Lemma 12.2, it follows that $(r_i + \ker \tilde{D}_i) \cap \tilde{f}_{i+1} k[\mathbf{x}] \neq \emptyset$. Since $\ker \tilde{D}_i = k[f_i, \tilde{f}_{i+1}]$, this implies that

$$S := (r_i + k[f_i]) \cap \tilde{f}_{i+1} k[\mathbf{x}] \neq \emptyset.$$

Note that S is contained in $\tilde{P} := k[f_i, r_i] \cap \tilde{f}_{i+1} k[\mathbf{x}]$. Since $\ker \tilde{D}_i = k[f_i, \tilde{f}_{i+1}]$ and $\tilde{D}_i(r_i) \neq 0$, we know by Lemma 4.2 (iii) that $\tilde{P}$ is a principal prime ideal of $k[f_i, r_i]$. As mentioned, $\tilde{q}_i$ is an irreducible element of $k[f_i, r_i]$ by (7.1). Since $\tilde{q}_i = f_{i-1}\tilde{f}_{i+1}$ belongs to $\tilde{f}_{i+1} k[\mathbf{x}]$, we conclude that $\tilde{P}$ is generated by $\tilde{q}_i$. Because S is contained in $\tilde{P}$, this implies that $\deg_{r_i} \tilde{q}_i \leq 1$. On the other hand, we see from (7.1) that

$$\deg_{r_i} \tilde{q}_i = \deg_z \tilde{h}_i(y, z) = \deg_z \tilde{\eta}_i(y, z) = a_i \geq 2,$$

a contradiction. Therefore, $\tilde{f}_{i+1}$ is a factor of $\tilde{g}_i$. In particular, we get $\operatorname{pl} \tilde{D}_i \neq \ker \tilde{D}_i$. Since $\tilde{D}_i$ is irreducible, this implies that $\operatorname{rank} \tilde{D}_i \geq 2$ by Lemma 12.1.

Assume that $\tilde{g}_i$ and $\tilde{D}_i(r_i)\tilde{f}_{i+1}^{-1}$ have a common factor $p \in k[\mathbf{x}] \setminus k$. Since $\tilde{D}_i(r_i)\tilde{f}_{i+1}^{-1}$ belongs to $k[f_i]$, and $k[f_i]$ is factorially closed in $k[\mathbf{x}]$ by Lemma 4.1, it follows that p belongs to $k[f_i] \setminus k$. Because f_i and $\tilde{f}_{i+1}$ are algebraically independent over k, we know that p and $\tilde{f}_{i+1}$ are algebraically independent over k. Therefore, we get $\operatorname{rank} \tilde{D}_i = 3$ by Proposition 12.4. $\square$

Now, we prove Theorem 1.6 (ii). Assume that $i = 2$ and $c := \lambda(y)$ belongs to $k^{\times}$. Then, we have $\tilde{D}_2(r_i) = c\tilde{f}_3$. Since $\tilde{g}_2$ is a factor of $\tilde{D}_2(r_i)$, and is divisible by $\tilde{f}_3$ by Lemma 13.1, it follows that $\tilde{g}_2 \approx \tilde{f}_3$. Hence, we get $\operatorname{pl} \tilde{D}_2 = (\tilde{f}_3)$. This implies that $\operatorname{pl} \tilde{D}_2 \neq \ker \tilde{D}_2$. Hence, we have $\operatorname{rank} \tilde{D}_2 \geq 2$ by Lemma 12.1. We show that $\tilde{f}_3$ is a coordinate of $k[\mathbf{x}]$ over k. Then, it follows that $\operatorname{rank} \tilde{D}_2 = 2$, and the proof is completed. Put $g = c^{-1}\mu(f_2, x_1)x_1^{-1}$. Then, g belongs to $k[x_1, f_2]$, since $\mu(y, z)$ is an element of $zk[y, z]$. Let $\bar{D}$ be the triangular derivation of $k[\mathbf{x}]$ defined by $\bar{D}(x_1) = 0$, $\bar{D}(x_2) = x_1$ and $\bar{D}(x_3) = \theta'(x_2)$. Then, we have $\bar{D}(f_2) = 0$, and so $\bar{D}(g) = 0$. Hence, $-g\bar{D}$ belongs to $\mathrm{LND}_k\, k[\mathbf{x}]$. Define $\bar{\sigma} = \exp(-g\bar{D})$. Then, we have

$\bar{\sigma}(x_1) = x_1$ and

$$\bar{\sigma}(x_2) = x_2 - gx_1 = x_2 - c^{-1}\mu(f_2, x_1) = c^{-1}r_2.$$

Since $f_2 = x_1x_3 - \theta(x_2)$ is equal to $\bar{\sigma}(f_2) = x_1\bar{\sigma}(x_3) - \theta(c^{-1}r_2)$, we know that

$$\bar{\sigma}(x_3) = x_3 + \big(\theta(c^{-1}r_2) - \theta(x_2)\big)x_1^{-1} = \big(x_1x_3 - \theta(x_2) - \theta(c^{-1}r_2)\big)x_1^{-1} = c^{-a_2}\tilde{f}_3.$$

Therefore, $\tilde{f}_3$ is a coordinate of $k[\mathbf{x}]$ over k. This proves Theorem 1.6 (ii).

Lemma 13.2. *Assume that $i \geq 3$. If $\lambda(0) \neq 0$, then f_i is a factor of $\tilde{g}_i$.*

Proof. Suppose to the contrary that f_i is not a factor of $\tilde{g}_i$. Then, $\operatorname{pl} \tilde{D}_i$ is not contained in $f_i \ker \tilde{D}_i$. Set $\mathfrak{p}_i = f_i k[\mathbf{x}]$. Then, we know by the last part of Lemma 12.2 that $(r_i + \ker \tilde{D}_i) \cap \mathfrak{p}_i$ is not empty. Since $\ker \tilde{D}_i = k[f_i, \tilde{f}_{i+1}]$, it follows that $(r_i + k[\tilde{f}_{i+1}]) \cap \mathfrak{p}_i$ is not empty. Take $\psi(z) \in k[z]$ such that $r_i - \psi(\tilde{f}_{i+1})$ belongs to $\mathfrak{p}_i$. Then, we have

$$\lambda(0)r - \mu(0, f_{i-1}) - \psi(\tilde{f}_{i+1}) \equiv r_i - \psi(\tilde{f}_{i+1}) \equiv 0 \pmod{\mathfrak{p}_i}. \tag{13.1}$$

Note that $(\lambda(0)r + k[f_{i-1}]) \cap \mathfrak{p}_i = \emptyset$ by the last part of Lemma 6.2, since $\lambda(0) \neq 0$ by assumption, and $a_i \geq 2$. Hence, we see from (13.1) that $\psi(z)$ does not belong to k. Define $\psi_1(z), \psi_2(z) \in k[z, z^{-1}]$ by

$$\psi_1(z) = \psi(z)^{a_i}z^{-1} \quad \text{and} \quad \psi_2(z) = \lambda(0)^{-1}\big(\mu(0, \psi_1(z)) + \psi(z)\big).$$

We show that $\eta_{i-1}(\psi_1(z), \psi_2(z)) = 0$. Since $k[\tilde{f}_{i+1}] \cap \mathfrak{p}_i = \{0\}$ by Lemma 4.2 (i), the image of $\tilde{f}_{i+1}$ in $k[\mathbf{x}]/\mathfrak{p}_i$ is transcendental over k. Hence, it suffices to verify that

$$h := \eta_{i-1}(\psi_1(\tilde{f}_{i+1}), \psi_2(\tilde{f}_{i+1}))$$

belongs to $\mathfrak{p}_i k[\mathbf{x}]_{\mathfrak{p}_i}$. By (7.5), we know that $f_{i-1}\tilde{f}_{i+1} = \tilde{q}_i$ is congruent to $r_i^{a_i}$ modulo $\mathfrak{p}_i$. Since $r_i \equiv \psi(\tilde{f}_{i+1}) \pmod{\mathfrak{p}_i}$ by the choice of $\psi(z)$, it follows that $f_{i-1}\tilde{f}_{i+1} \equiv \psi(\tilde{f}_{i+1})^{a_i} \pmod{\mathfrak{p}_i}$. Hence, $\psi_1(\tilde{f}_{i+1}) = \psi(\tilde{f}_{i+1})^{a_i}\tilde{f}_{i+1}^{-1}$ is congruent to f_{i-1} modulo $\mathfrak{p}_i k[\mathbf{x}]_{\mathfrak{p}_i}$. This implies that

$$\psi_2(\tilde{f}_{i+1}) \equiv \lambda(0)^{-1}\big(\mu(0, f_{i-1}) + \psi(\tilde{f}_{i+1})\big) \equiv r \pmod{\mathfrak{p}_i k[\mathbf{x}]_{\mathfrak{p}_i}}$$

by (13.1). Thus, h is congruent to $\eta_{i-1}(f_{i-1}, r) = f_{i-2}f_i$ modulo $\mathfrak{p}_i k[\mathbf{x}]_{\mathfrak{p}_i}$, and therefore belongs to $\mathfrak{p}_i k[\mathbf{x}]_{\mathfrak{p}_i}$. This proves that $\eta_{i-1}(\psi_1(z), \psi_2(z)) = 0$.

Put $l_j = \deg_z \psi_j(z)$ for $j = 1, 2$. Then, we claim that

$$t_{i-1}l_1 = a_{i-1}l_2. \tag{13.2}$$

In fact, if $t_{i-1}l_1 \neq a_{i-1}l_2$, then it follows from Lemma 5.4 that

$$\deg_z \eta_{i-1}(\psi_1(z), \psi_2(z)) = \max\{t_{i-1}l_1, a_{i-1}l_2\} > -\infty,$$

a contradiction. Since $\psi(z)$ does not belong to k as mentioned, we have $l := \deg_z \psi(z) \geq 1$. Since $a_i \geq 2$, it follows that

$$l_1 = \deg_z \psi_1(z) = \psi(z)^{a_i}z^{-1} = a_il - 1 \geq 1. \tag{13.3}$$

Put $m = \deg_z \mu(0, z)$. Then, we have

$$l_2 = \deg_z \psi_2(z) = \deg_z\big(\mu(0, \psi_1(z)) + \psi(z)\big) \leq \max\{l_1m, l\}, \tag{13.4}$$

in which the equality holds when $l_1m \neq l$.

First, assume that $\mu(0,z)=0$. Then, we have $\psi_2(z)\approx\psi(z)$, and so $l_2=l$. Hence, it follows from (13.3) and (1.7) that

$$t_{i-1}l_1-a_{i-1}l_2=t_{i-1}(a_il-1)-a_{i-1}l=(t_{i-1}a_i-a_{i-1})l-t_{i-1}=a_{i+1}l-t_{i-1}.$$

Since $l\geq 1$, and $a_{i+1}>t_{i+1}$ by Lemma 3.1 (iii), we know that $a_{i+1}l-t_{i-1}>0$. Thus, we get a contradiction to (13.2).

Next, assume that $\mu(0,z)\neq 0$. Then, $\mu(0,z)$ belongs to $zk[z]\setminus\{0\}$, since $\mu(y,z)$ is an element of $zk[y,z]$. Hence, we have $m=\deg_z\mu(0,z)\geq 1$.

Assume that $i=3$. If $l_2=l_1m$, then we have $t_2l_1=a_2l_2=a_2l_1m$ by (13.2). Hence, we get $m=t_2/a_2=t_0/(t_0-1)$. Since $t_0\geq 3$, it follows that m is not an integer, a contradiction.

Assume that $l_2\neq l_1m$. Then, we have $l_1m\leq l$ in view of (13.4) and the note following it. By (13.3), it follows that

$$l\geq l_1m=(a_3l-1)m=\big((a_3-1)l+(l-1)\big)m.$$

Since $a_3\geq 2$, $l\geq 1$ and $m\geq 1$, this implies that $l=m=l_1=1$ and $a_3=2$. Since $l_1m\leq l$, we have $l_2\leq l$ by (13.4), and so $l_2\leq 1$. By the assumption that $l_2\neq l_1m$, it follows that $l_2\leq 0$. Hence, we get $3\leq t_2=t_2l_1=a_2l_2\leq 0$ by (13.2), a contradiction.

Finally, assume that $i\geq 4$. Then, we have $a_i\geq 3$ by (i) and (ii) of Lemma 3.1. Since $l\geq 1$, it follows that $l_1=a_il-1>l$ by (13.3). Hence, we get $l_1m>l$. This implies that $l_2=l_1m$ by (13.4) and the note following it. Since $i-1\geq 3$, we have $t_{i-1}<a_{i-1}$ by Lemma 3.1 (iii). Thus, we know that $t_{i-1}l_1<a_{i-1}l_1\leq a_{i-1}l_1m=a_{i-1}l_2$, a contradiction to (13.2). Therefore, f_i is a factor of $\tilde{g}_i$. □

If $i\geq 3$, then f_i is a factor of $\tilde{D}_i(r_i)$, and hence a factor of $\tilde{D}_i(r_i)\tilde{f}_{i+1}^{-1}$. By Lemma 13.2, it follows that f_i is a common factor of $\tilde{g}_i$ and $\tilde{D}_i(r_i)\tilde{f}_{i+1}^{-1}$ if $i\geq 3$ and $\lambda(0)\neq 0$. If this is the case, then we have $\operatorname{rank}\tilde{D}_i=3$ due to the last part of Lemma 13.1. This proves Theorem 1.6 (i) when $\lambda(0)\neq 0$. To complete the proof of Theorem 1.6, we have only to prove (i) when $\lambda(0)=0$, and (iii). Therefore, we assume that $\lambda(y)$ is not an element of k in what follows.

Let $\alpha\in\bar{k}$ be a root of $\lambda(y)$, and $\lambda_\alpha(y)$ the minimal polynomial of α over k. We define $m\in\mathbf{N}$ to be the maximal number such that $\lambda_\alpha(y)^m$ is a factor of $\lambda(y)$ if $i=2$, and of $y\lambda(y)$ if $i\geq 3$. Then, we have the following lemma.

Lemma 13.3. *If $a_i\geq 3$ or $\deg_z\mu(\alpha,z)\geq 2$, then $\lambda_\alpha(f_i)^m$ is a factor of $\tilde{g}_i$.*

Proof. Since $p:=\lambda_\alpha(f_i)$ is an irreducible element of $k[f_i]$, and $k[f_i]$ is factorially closed in $k[\mathbf{x}]$ by Lemma 4.1, it follows that p is an irreducible element of $k[\mathbf{x}]$. Since $\tilde{D}_i(r_i)$ is divisible by $\lambda(f_i)$ if $i=2$, and by $\lambda(f_i)f_i$ if $i\geq 3$, we have $v_p(\tilde{D}_i(r_i))\geq m$ by the definition of m. We show that $(r_i+\ker\tilde{D}_i)\cap pk[\mathbf{x}]=\emptyset$. Then, it follows that $v_p(\tilde{g}_i)=v_p(\tilde{D}_i(r_i))\geq m$ by Lemma 12.2, and the lemma is proved.

Suppose to the contrary that $(r_i+\ker\tilde{D}_i)\cap pk[\mathbf{x}]$ is not empty. Since $\ker\tilde{D}_i=k[f_i,\tilde{f}_{i+1}]$, we may find $\nu(y,z)\in k[y,z]$ such that $r_i+\nu(f_i,\tilde{f}_{i+1})$ belongs to $pk[\mathbf{x}]$. Then, $r_i+\nu(f_i,\tilde{f}_{i+1})$ belongs to $\mathfrak{p}:=(f_i-\alpha)\bar{k}[\mathbf{x}]$. Put $\bar{\mu}(z)=-\mu(\alpha,z)$ and $\bar{\nu}(z)=\nu(\alpha,z)$. Then, $\bar{\mu}(z)$ belongs to $z\bar{k}[z]$, since $\mu(y,z)$ is an element of $zk[y,z]$. We show that $\psi(z):=\bar{\mu}(z)+\bar{\nu}\big(\bar{\mu}(z)^{a_i}z^{-1}\big)=$

0. As remarked after Lemma 4.1, D_i extends to an element of $\mathrm{LND}_{\bar{k}}\,\bar{k}[\mathbf{x}]$ with kernel $\bar{k}[f_{i-1}, f_i - \alpha]$. Hence, $\mathfrak{p}$ is a prime ideal of $\bar{k}[\mathbf{x}]$ by Lemma 4.1. Since $\bar{k}[f_{i-1}] \cap \mathfrak{p} = \{0\}$ by Lemma 4.2 (i), the image of f_{i-1} in $\bar{k}[\mathbf{x}]/\mathfrak{p}$ is transcendental over $\bar{k}$. Hence, it suffices to verify that $\psi(f_{i-1}) \equiv 0 \pmod{\mathfrak{p}}$. Note that

$$f_{i-1}\tilde{f}_{i+1} = \tilde{\eta}_i(f_i, r_i\lambda(f_i)^{-1})\lambda(f_i)^{a_i}$$

and $\tilde{\eta}_i(y, z)$ is a monic polynomial in z of degree a_i over $k[y]$. Since $\lambda(\alpha) = 0$, we see that $\tilde{\eta}_i(f_i, r_i\lambda(f_i)^{-1})\lambda(f_i)^{a_i} \equiv r_i^{a_i} \pmod{\mathfrak{p}}$, while $r_i = \lambda(f_i)\tilde{r} - \mu(f_i, f_{i-1})$ is congruent to $\lambda(\alpha)\tilde{r} - \mu(\alpha, f_{i-1}) = \bar{\mu}(f_{i-1})$ modulo $\mathfrak{p}$. Hence, it follows that $f_{i-1}\tilde{f}_{i+1} \equiv \bar{\mu}(f_{i-1})^{a_i} \pmod{\mathfrak{p}}$, and so $\tilde{f}_{i+1} \equiv \bar{\mu}(f_{i-1})^{a_i} f_{i-1}^{-1}$ $\pmod{\mathfrak{p}}$. Thus, we get

$$\begin{aligned}\psi(f_{i-1}) &= \bar{\mu}(f_{i-1}) + \bar{\nu}(\bar{\mu}(f_{i-1})^{a_i} f_{i-1}^{-1})\\ &\equiv r_i + \bar{\nu}(\tilde{f}_{i+1}) \equiv r_i + \nu(f_i, \tilde{f}_{i+1}) \equiv 0 \pmod{\mathfrak{p}}\end{aligned}$$

by the choice of $\nu(y, z)$. This proves that $\psi(z) = 0$. Hence, we know that $\bar{\mu}(z) = -\bar{\nu}(\bar{\mu}(z)^{a_i} z^{-1})$. From this, we obtain

$$l := \deg_z \bar{\mu}(z) = \deg_z \bar{\nu}(\bar{\mu}(z)^{a_i} z^{-1}) = (a_i l - 1)\deg_z \bar{\nu}(z). \tag{13.5}$$

Note that $a_i \geq 3$ or $l \geq 2$ by assumption. We claim that $l \geq 1$. In fact, $\bar{\mu}(z) = -\mu(\alpha, z)$ belongs to $z\bar{k}[z]$, and is nonzero by the assumption that $\lambda(y)$ and $\mu(y, z)$ have no common factor. Since $a_i \geq 2$, it follows from (13.5) that $\deg_z \bar{\nu}(z) = 1$, $a_i = 2$ and $l = 1$, a contradiction. Therefore, $(r_i + \ker \tilde{D}_i) \cap pk[\mathbf{x}]$ is empty. $\square$

In the situation of Lemma 13.3, we have $\operatorname{rank} \tilde{D}_i = 3$ due to the last part of Lemma 13.1. In particular, we have $\operatorname{rank} \tilde{D}_i = 3$ if $a_i \geq 3$. We note that $a_i \geq 3$ in the following cases:

(I) $i = 2$ and $t_0 \geq 4$.

(II) $i \geq 3$ and $(t_0, t_1, i) \neq (4, 1, 3)$.

Actually, we have $a_2 = t_0 - 1 \geq 3$ in the case of (I). If $i \geq 3$, then we have $t_0 \geq 3$ and $(t_0, t_1) \neq (3, 1)$ by assumption. Hence, we know that $a_3 = t_1(t_0 - 1) - 1 \geq 3$ if $(t_0, t_1) \neq (4, 1)$. If $i \geq 4$, then it follows that $a_i \geq 3$ by (i) and (ii) of Lemma 3.1.

Similarly, we have $\operatorname{rank} \tilde{D}_i = 3$ if $\deg_z \mu(\alpha, z) \geq 2$ for some $\alpha \in \bar{k}$ with $\lambda(\alpha) = 0$. This condition is equivalent to the following condition:

(III) $\mu_j(y)$ does not belong to $\sqrt{(\lambda(y))}$ for some $j \geq 2$.

From (I) and (III), we get the first part of Theorem 1.6 (iii). From (II), (III) and the discussion after Lemma 13.2, we see that Theorem 1.6 (i) is proved except for the case where $\lambda(0) = 0$, $(t_0, t_1, i) = (4, 1, 3)$ and $\mu_j(y)$ belongs to $\sqrt{(\lambda(y))}$ for every $j \geq 2$. To complete the proof of Theorem 1.6, it suffices to prove (i) in this exceptional case, and the last part of (iii). Therefore, we are reduced to proving the following statements under the assumption that $\mu_j(y)$ belongs to $\sqrt{(\lambda(y))}$ for every $j \geq 2$:

(1) If $\lambda(0) = 0$ and $(t_0, t_1) = (4, 1)$, then we have $\operatorname{rank} \tilde{D}_3 = 3$.

(2) If $t_0 = 3$, then we have $\tilde{g}_2 \approx \tilde{f}_3$.

We remark that $\lambda(y)$ and $\mu(y, z) - \mu_1(y)z$ have a non-constant common factor under the assumption above, since we are assuming that $\lambda(y)$ is not

an element of k. Hence, $\lambda(y)$ and $\mu_1(y)$ have no common factor, since so do $\lambda(y)$ and $\mu(y,z)$.

First, we prove (1). By Lemma 4.1, f_3 and $\tilde{f}_4$ are irreducible elements of $k[\mathbf{x}]$ with $f_3 \not\approx \tilde{f}_4$. Since $\tilde{D}_3(r_3) = \lambda(f_3)f_3\tilde{f}_4$, and $\lambda(0) = 0$ by assumption, we have

$$a := v_{f_3}(\tilde{D}_3(r_3)) \geq 2 \quad \text{and} \quad v_{f_3}(\lambda(f_3)) = a - 1,$$

and so $v_{f_3}(\lambda(f_3)^2) = 2(a-1) \geq a$. Hence, we get $\lambda(f_3)^2 \equiv 0 \pmod{f_3^a k[\mathbf{x}]}$. Set

$$f = \big(\mu(f_3,f_2)^2 - 2\lambda(f_3)\mu(f_3,f_2)r\big)f_2^{-1}.$$

Then, f belongs to $k[\mathbf{x}]$, since $\eta(y,z)$ is an element of $zk[y,z]$. We show that $\tilde{f}_4 \equiv f \pmod{f_3^a k[\mathbf{x}]}$. Since $(t_0,t_1) = (4,1)$, we have $a_3 = t_1(t_0-1)-1 = 2$ and $\tilde{\eta}_3(y,z) = \eta_3(y,z) = y + z^2$. Hence, we get

$$f_2\tilde{f}_4 = \tilde{\eta}_3(f_3, r_3\lambda(f_3)^{-1})\lambda(f_3)^2 = f_3\lambda(f_3)^2 + r_3^2.$$

From this, we see that $f_2\tilde{f}_4$ is congruent to

$$r_3^2 = (\lambda(f_3)\tilde{r} - \mu(f_3,f_2))^2 = \lambda(f_3)^2r^2 - 2\lambda(f_3)r\mu(f_3,f_2) + \mu(f_3,f_2)^2,$$

and hence to f_2f modulo $f_3^a k[\mathbf{x}]$. Since f_2 and f_3 have no common factor, it follows that $\tilde{f}_4 \equiv f \pmod{f_3^a k[\mathbf{x}]}$.

To conclude that $\operatorname{rank} D = 3$, it suffices to show that f_3 is a factor of $\tilde{g}_3$ thanks to the last part of Lemma 13.1, since f_3 is a factor of $\tilde{D}_3(r_3)\tilde{f}_4^{-1} = \lambda(f_3)f_3$. Suppose to the contrary that $v_{f_3}(\tilde{g}_3) = 0$. Then, $(r_3 + \ker \tilde{D}_3) \cap f_3^a k[\mathbf{x}]$ is not empty by Lemma 12.2. Since $\ker \tilde{D}_3 = k[f_3, \tilde{f}_4]$, we may find $\nu(y,z) \in k[y,z]$ such that $r_3 + \nu(f_3,\tilde{f}_4)$ belongs to $f_3^a k[\mathbf{x}]$. Since $\tilde{f}_4 \equiv f$ $\pmod{f_3^a k[\mathbf{x}]}$, it follows that $r_3 + \nu(f_3, f)$ belongs to $f_3^a k[\mathbf{x}]$. We note that r_3 and f are linear polynomials in r over $k[f_2,f_3]$ whose leading coefficients are multiples of $\lambda(f_3)$. Here, r, f_2 and f_3 are algebraically independent over k because

$$df_3 \wedge df_2 \wedge dr = D_3(r)dx_1 \wedge dx_2 \wedge dx_3 \neq 0.$$

Hence, we may uniquely write

$$r_3 + \nu(f_3, f) = \sum_{i,j} \psi_{i,j}(f_3)r^i f_2^j,$$

where $\psi_{i,j}(y)$ is an element of $\lambda(y)^i k[y]$ for each i and j. We show that $v_y(\psi_{1,0}(y))$ or $v_y(\psi_{0,1}(y))$ is less than a. Let (r,f_2) be the ideal of $k[r,f_2,f_3]$ generated by r and f_2. Then, we have $\mu(f_3,f_2) \equiv \mu_1(f_3)f_2 \pmod{(r,f_2)^2}$. Hence, we get

$$r_3 \equiv \lambda(f_3)r - \mu_1(f_3)f_2 \pmod{(r,f_2)^2}$$
$$f \equiv \mu_1(f_3)^2f_2 - 2\lambda(f_3)\mu_1(f_3)r \pmod{(r,f_2)^2}.$$

Write $\nu(y,z) = \sum_{j\geq 0} \nu_j(y)z^j$, where $\nu_j(y) \in k[y]$ for each j. Then, we have

$$\nu(f_3,f) \equiv \nu_1(f_3)f + \nu_0(f_3) \pmod{(r,f_2)^2},$$

since f belongs to (r,f_2). Hence, $r_3 + \nu(f_3,f) - \nu_0(f_3)$ is congruent to

$$\begin{aligned}&\big(\lambda(f_3)r - \mu_1(f_3)f_2\big) + \nu_1(f_3)\big(\mu_1(f_3)^2f_2 - 2\lambda(f_3)\mu_1(f_3)r\big)\\ &\quad = \big(1 - 2\nu_1(f_3)\mu_1(f_3)\big)\lambda(f_3)r + \mu_1(f_3)\big(\nu_1(f_3)\mu_1(f_3) - 1\big)f_2\end{aligned}$$

modulo $(r, f_2)^2$. From this, we see that

$$\psi_{1,0}(y) = \bigl(1 - 2\nu_1(y)\mu_1(y)\bigr)\lambda(y) \quad \text{and} \quad \psi_{0,1}(y) = \mu_1(y)\bigl(\nu_1(y)\mu_1(y) - 1\bigr).$$

Now, assume that $v(\psi_{1,0}(y)) \geq a$. Then, $1 - 2\nu_1(y)\mu_1(y)$ belongs to $yk[y]$, since $v_y(\lambda(y)) = v_{f_3}(\lambda(f_3)) = a-1$. This implies that $\nu_1(y)\mu_1(y)-1$ does not belong to $yk[y]$. Since $\lambda(0) = 0$ by assumption, and $\lambda(y)$ and $\mu_1(y)$ have no common factor as mentioned, we know that $\mu_1(y)$ does not belong to $yk[y]$. Hence, $\psi_{0,1}(y)$ does not belong to $yk[y]$. Thus, we get $v_y(\psi_{0,1}(y)) = 0 < a$. Therefore, $v_y(\psi_{1,0}(y))$ or $v_y(\psi_{0,1}(y))$ is less than a. Consequently, we have

$$b := \min\{v_y(\psi_{i,j}(y)) \mid i, j\} < a.$$

Since $r_3 + \nu(f_3, f)$ belongs to $f_3^a k[\mathbf{x}]$, it follows that $g := (r_3 + \nu(f_3, f))f_3^{-b}$ belongs to $f_3 k[\mathbf{x}]$. Set $\bar{\psi}_{i,j}(y) = \psi_{i,j}(y)y^{-b}$ for each i and j. Then, $\bar{\psi}_{i,j}(y)$ belongs to $k[y]$ for every i and j, and $\bar{\psi}_{i,j}(0) \neq 0$ for some i and j. Hence, $h := \sum_{i,j} \bar{\psi}_{i,j}(0) r^i f_2^j$ is a nonzero element of $k[r, f_2]$. We note that $\deg_r h \leq 1$. In fact, we have

$$v_y(\psi_{i,j}(y)) \geq v_y(\lambda(y)^2) = v_{f_3}(\lambda(f_3)^2) \geq a$$

if $i \geq 2$, since $\psi_{i,j}(y)$ is an element of $\lambda(y)^i k[y]$. Since $g = \sum_{i,j} \bar{\psi}_{i,j}(f_3) r^i f_2^j$, we have $h \equiv g \pmod{f_3 k[\mathbf{x}]}$. Because g belongs to $f_3 k[\mathbf{x}]$, it follows that h belongs to $f_3 k[\mathbf{x}]$. Thus, h belongs to $k[f_2, r] \cap f_3 k[\mathbf{x}] \setminus \{0\}$. By Lemma 6.2, we have $k[f_2, r] \cap f_3 k[\mathbf{x}] = q_2 k[f_2, r]$. Therefore, we get $\deg_r h \geq \deg_r q_2 = a_2 = t_0 - 1 = 3$, a contradiction. This proves that $\operatorname{rank} \tilde{D}_3 = 3$, and thereby completing the proof of (1).

Next, we prove (2). Suppose to the contrary that $\tilde{g}_2 \not\approx \tilde{f}_3$. Then, $\tilde{g}_2 \tilde{f}_3^{-1}$ belongs to $k[\mathbf{x}] \setminus k$, since $\tilde{f}_3$ is a factor of $\tilde{g}_2$ by Lemma 13.1. Let p be a factor of $\tilde{g}_2 \tilde{f}_3^{-1}$ which is an irreducible element of $k[\mathbf{x}]$. Since $\tilde{g}_2$ is a factor of $\lambda(f_2)\tilde{f}_3$, we know that $\tilde{g}_2 \tilde{f}_3^{-1}$ is a factor of $\lambda(f_2)$. Hence, p is a factor of $\lambda(f_2)$. Since $k[f_2]$ is factorially closed in $k[\mathbf{x}]$ by Lemma 4.1, it follows that p belongs to $k[f_2]$. By Lemma 4.1, $\tilde{f}_3$ is an irreducible element of $k[\mathbf{x}]$, and does not belong to $k[f_2]$. Hence, p does not divide $\tilde{f}_3$. Thus, we get

$$a := v_p\bigl(\tilde{D}_2(r_2)\bigr) = v_p\bigl(\lambda(f_2)\tilde{f}_3\bigr) = v_p\bigl(\lambda(f_2)\bigr) \geq 1. \tag{13.6}$$

Set $\mathfrak{q} = p^a k[\mathbf{x}]$. Then, we have $(r_2 + \ker \tilde{D}_2) \cap \mathfrak{q} = \emptyset$ by Lemma 12.2, since $v_p(\tilde{g}_2) = v_p(\tilde{g}_2 \tilde{f}_3^{-1}) \geq 1$ by the choice of p.

We define $\tilde{f} = \mu(f_2, x_1)^2 x_1^{-1}$. Then, $\tilde{f}$ belongs to $x_1 k[x_1, f_2]$. We show that $\tilde{f}_3 \equiv \tilde{f} \pmod{\mathfrak{q}}$. By (13.6), we have $\lambda(f_2) \equiv 0 \pmod{\mathfrak{q}}$. Since $\tilde{\eta}_2(y, z) = y + \alpha_1^0 + \alpha_2^0 z + z^2$, we see that $x_1 \tilde{f}_3 = \tilde{\eta}_2(f_2, r_2\lambda(f_2)^{-1})\lambda(f_2)^2$ is congruent to r_2^2 modulo $\mathfrak{q}$. On the other hand, $r_2 = \lambda(f_2)x_2 - \mu(f_2, x_1)$ is congruent to $-\mu(f_2, x_1)$ modulo $\mathfrak{q}$. Hence, we get $x_1 \tilde{f}_3 \equiv \mu(f_2, x_1)^2 \pmod{\mathfrak{q}}$. Since p is an element of $k[f_2]$, we have $x_1 \not\approx p$. Therefore, $\tilde{f}_3$ is congruent to $\mu(f_2, x_1)^2 x_1^{-1} = \tilde{f}$ modulo $\mathfrak{q}$.

The following is a key claim.

Claim 13.4. *There exists $h \in \mu(f_2, x_1) + k[f_2, \tilde{f}]$ such that $v_p(h) \geq a$.*

Assuming this claim, we can complete the proof of (2) as follows. By the claim, there exists $\nu(y, z) \in k[y, z]$ such that $\mu(f_2, x_1) + \nu(f_2, \tilde{f})$ belongs

to $\mathfrak{q}$. Then, we have

$$r_2 - \nu(f_2, \tilde{f}_3) \equiv -\mu(f_2, x_1) - \nu(f_2, \tilde{f}) \equiv 0 \pmod{\mathfrak{q}},$$

since $r_2 \equiv -\mu(f_2, x_1) \pmod{\mathfrak{q}}$ and $\tilde{f}_3 \equiv \tilde{f} \pmod{\mathfrak{q}}$. Hence, $r_2 - \nu(f_2, \tilde{f}_3)$ belongs to $\mathfrak{q}$. This contradicts that $(r_2 + \ker \tilde{D}_2) \cap \mathfrak{q} = \emptyset$, and thereby proving $\tilde{g}_2 \approx \tilde{f}_3$.

Let us prove Claim 13.4. Note that $\mu(f_2, x_1) + \tilde{f}k[f_2, \tilde{f}]$ is contained in $x_1 k[x_1, f_2]$, since $\mu(f_2, x_1)$ and $\tilde{f}$ belong to $x_1 k[x_1, f_2]$. Hence, each element of $\mu(f_2, x_1) + \tilde{f}k[f_2, \tilde{f}]$ is written as

$$h = \sum_{j \geq 1} h_j x_1^j, \quad \text{where} \quad h_j \in k[f_2] \quad \text{for each} \quad j \geq 1.$$

By assumption, $\mu_j(y)$ belongs to $\sqrt{(\lambda(y))}$ for each $j \geq 2$. Since p is a factor of $\lambda(f_2)$, and is an irreducible element of $k[\mathbf{x}]$, this implies that p divides $\mu_j(f_2)$ for each $j \geq 2$. Hence, we have

$$b := \min \left\{ \frac{v_p(\mu_j(f_2))}{j-1} \;\middle|\; j \geq 2 \right\} > 0.$$

Let $\mathcal{S}$ be the set of $h \in \mu(f_2, x_1) + \tilde{f}k[f_2, \tilde{f}]$ such that

$$v_p(h_j) \geq (j-1)b \quad \text{for each} \quad j \geq 2. \tag{13.7}$$

Then, $\mu(f_2, x_1)$ belongs to $\mathcal{S}$, since

$$v_p(\mu_j(f_2)) \geq (j-1)b \quad \text{for each} \quad j \geq 2 \tag{13.8}$$

by the definition of b. Hence, $\mathcal{S}$ is not empty. To prove Claim 13.4, it suffices to verify that $v_p(h) \geq a$ for some $h \in \mathcal{S}$. Suppose to the contrary that $v_p(h) < a$ for all $h \in \mathcal{S}$. Then, we can find

$$l(h) := \min\{l \in \mathbf{N} \mid v_p(h_l) < a\}$$

for each $h \in \mathcal{S}$. Actually, if $v_p(h_l) \geq a$ for every $l \in \mathbf{N}$ for $f \in \mathcal{S}$, then we have $v_p(h) \geq a$. By (13.7), we have

$$\bigl(l(h) - 1\bigr)b \leq v_p(h_{l(h)}) < a \tag{13.9}$$

for each $h \in \mathcal{S}$. Hence, we can find $l := \max\{l(h) \mid h \in \mathcal{S}\}$. Take $h \in \mathcal{S}$ and $u(y) \in k[y]$ such that $l = l(h)$ and $h_l = u(f_2)$. Since $\lambda(y)$ and $\mu_1(y)$ have no common factor, the same holds for $\lambda(y)$ and $\mu_1(y)^{2l}$. Hence, we may find $u_1(y), u_2(y) \in k[y]$ such that

$$u_1(y)\lambda(y) + u_2(y)\mu_1(y)^{2l} = u(y). \tag{13.10}$$

Since $k[f_2]$ is factorially closed in $k[\mathbf{x}]$, we see that $\lambda(f_2)$ and $\mu_1(f_2)$ have no common factor. Since p is a factor of $\lambda(f_2)$, it follows that $v_p(\mu_1(f_2)) = 0$. Hence, we get

$$\begin{aligned} v_p\bigl(u_2(f_2)\bigr) &= v_p\bigl(u_2(f_2)\mu_1(f_2)^{2l}\bigr) = v_p\bigl(u(f_2) - u_1(f_2)\lambda(f_2)\bigr) \\ &= v_p\bigl(h_l - u_1(f_2)\lambda(f_2)\bigr) \geq \min\{v_p\bigl(h_l\bigr), v_p\bigl(u_1(f_2)\lambda(f_2)\bigr)\} \\ &\geq \min\{(l-1)b, a\} = (l-1)b \end{aligned} \tag{13.11}$$

in view of (13.6) and (13.9).

Write

$$\mu(f_2, x_1)^{2l} = \left(\sum_{j\geq 1} \mu_j(f_2)x_1^j\right)^{2l} = \sum_{j\geq l} g_j x_1^{j+l},$$

where g_j is the sum of $\prod_{t=1}^{2l} \mu_{j_t}(f_2)$ for $j_1, \ldots, j_{2l} \in \mathbf{N}$ such that $\sum_{t=1}^{2l} j_t = j + l$ for each $j \geq l$. Then, we have $g_l = \mu_1(f_2)^{2l}$, and $v_p(g_j) \geq (j-l)b$ for each $j \geq l$, since

$$v_p\left(\prod_{t=1}^{2l} \mu_{j_t}(f_2)\right) = \sum_{t=1}^{2l} v_p\bigl(\mu_{j_t}(f_2)\bigr) \geq \sum_{t=1}^{2l}(j_t - 1)b = (j-l)b$$

by (13.8). Hence, it follows from (13.11) that

$$v_p\bigl(u_2(f_2)g_j\bigr) = v\bigl(u_2(f_2)\bigr) + v_p(g_j) \geq (l-1)b + (j-l)b = (j-1)b \tag{13.12}$$

for each $j \geq l$.

Now, consider the polynomial

$$h' := h - u_2(f_2)\tilde{f}^l = h - u_2(f_2)\mu(f_2, x_1)^{2l}x_1^{-l} = \sum_{j\geq 1}\bigl(h_j - u_2(f_2)g_j\bigr)x_1^j,$$

where $g_j := 0$ for $1 \leq j < l$. Since h is an element of $\mu(f_2, x_1) + \tilde{f}k[f_2, \tilde{f}]$, and $u_2(f_2)\tilde{f}^l$ belongs to $\tilde{f}k[f_2, \tilde{f}]$, it follows that h' belongs to $\mu(f_2, x_1) + \tilde{f}k[f_2, \tilde{f}]$. By (13.7) and (13.12), we have

$$v_p\bigl(h_j - u_2(f_2)g_j\bigr) \geq \min\{v_p(h_j), v_p\bigl(u_2(f_2)g_j\bigr)\} \geq (j-1)b$$

for each $j \geq 1$. Hence, h' belongs to $\mathcal{S}$. For each $1 \leq j < l$, we have

$$v_p\bigl(h_j - u_2(f_2)g_j\bigr) = v_p(h_j) \geq a$$

by the definition of $l = l(h)$. From (13.6) and (13.10), we see that

$$v_p\bigl(h_l - u_2(f_2)g_l\bigr) = v_p\bigl(u(f_2) - u_2(f_2)\mu_1(f_2)^{2l}\bigr) = v_p\bigl(u_1(f_2)\lambda(f_2)\bigr) \geq v_p(\lambda(f_2)) = a.$$

Hence, we get $l(h') > l$. This contradicts the maximality of l. Thus, we have proved Claim 13.4, and thereby proving (2). This completes the proof of Theorem 1.6.

14. Further local slice constructions

In this section, we discuss how to get more examples of elements of $\mathrm{LND}_k\, k[\mathbf{x}]$. First, we note that r may be replaced with a polynomial of the more general form $x_1x_2x_3 - \psi(x_1, x_2)$ in the construction of $(f_i)_{i=0}^{\infty}$. Here, $\psi(x_1, x_2) \in k[x_1, x_2]$ is such that $x_1x_2x_3 - \psi(0, x_2) - \psi(x_1, 0) = r$. In fact, since $\psi(0, 0) = 0$, we can define $\tau \in \mathrm{Aut}(k[\mathbf{x}]/k[x_1, x_2])$ by

$$\tau(x_3) = x_3 + (\psi(x_1, x_2) - \psi(0, x_2) - \psi(x_1, 0))x_1^{-1}x_2^{-1}.$$

Then, we have $\tau(f_0) = f_0$, $\tau(f_1) = f_1$ and $\tau\bigl(x_1x_2x_3 - \psi(x_1, x_2)\bigr) = r$. Therefore, we get $\bigl(\tau^{-1}(f_i)\bigr)_{i=0}^{\infty}$ instead of $(f_i)_{i=0}^{\infty}$ by this construction.

In the construction of $\tilde{f}_{i+1}$ for $i \geq 3$, we may interchange the role of f_{i-1} and f_{i+1}. Namely, when $i \neq \max I$, we consider

$$s_i := \lambda(f_i)r - \mu(f_i, f_{i+1}) \quad \text{and} \quad \tilde{g}_{i-1} := \tilde{h}_i(f_i, s_i)\lambda(f_i)^{a_i} f_{i+1}^{-1}$$

instead of r_i and $\tilde{f}_{i+1}$. Since $i \neq \max I$, we have $\ker D_i = k[f_i, f_{i+1}]$, and D_i is irreducible due to Theorem 1.1 (i). Hence, $-D_i = \Delta_{(f_i,f_{i+1})}$ satisfies (LSC1) for $f = f_i$ and $g = f_{i+1}$. By Lemma 4.1, $\mathfrak{p}_{i+1} := f_{i+1}k[\mathbf{x}]$ is a prime ideal of $k[\mathbf{x}]$. We claim that s_i does not belong to $\mathfrak{p}_{i+1}$. Actually, $\lambda(f_i)$ and r do not belong to $\mathfrak{p}_{i+1}$ by Lemma 4.2 (i) and by (1) of Proposition 6.1, while $\mu(f_i, f_{i+1})$ belongs to $\mathfrak{p}_{i+1}$ since $\mu(y, z)$ belongs to $zk[y, z]$. Since $D_i(f_i) = D_i(f_{i+1}) = 0$, we get $-D_i(s_i) = -\lambda(f_i)D_i(r) = -\lambda(f_i)f_if_{i+1}$. Thus, $-D_i$ satisfies (LSC2) for $s = s_i$ and $F = -\lambda(f_i)f_i$. By the irreducibility of $\tilde{h}_i(x_1, x_2)$, it follows that $f_{i+1}\tilde{g}_{i-1} = \tilde{h}_i(f_i, s_i)$ is an irreducible element of $k[f_i, s_i]$. Since $s_i \equiv \lambda(f_i)r \pmod{\mathfrak{p}_{i+1}}$, we see that $\tilde{h}_i(f_i, s_i)$ is congruent to $\eta_i(f_i, r)\lambda(f_i)^{a_i} = f_{i-1}f_{i+1}\lambda(f_i)^{a_i}$ modulo $\mathfrak{p}_{i+1}$. Hence, $\tilde{g}_{i-1}$ belongs to $k[\mathbf{x}]$. Therefore, we know by (a) and (b) of Theorem 4.3 that $\Delta_{(f_i,\tilde{g}_{i-1})}$ is locally nilpotent and $\Delta_{(f_i,\tilde{g}_{i-1})}(s_i) = \tilde{g}_{i-1}\lambda(f_i)f_i$.

Next, let $\Delta_{(f,h)}$ be an element of $\mathrm{LND}_k\, k[\mathbf{x}]$ obtained from a data (f, g, s) by a local slice construction. We consider when $\Delta_{(f,h)}$ has rank three. In view of Theorems 1.3 and 1.6, we see that there exist many examples in which $\Delta_{(f,g)}$ and $\Delta_{(f,h)}$ both have rank three. We are interested in the case where $\operatorname{rank}\Delta_{(f,g)} \leq 2$ and $\operatorname{rank}\Delta_{(f,h)} = 3$.

Proposition 14.1. *Assume that* $\operatorname{rank}\Delta_{(f,g)} \leq 2$ *and* $\operatorname{rank}\Delta_{(f,h)} = 3$. *Then, we have* $\operatorname{rank}\Delta_{(f,g)} = 2$, $\operatorname{pl}\Delta_{(f,g)} = (g)$, *and g is a coordinate of $k[\mathbf{x}]$ over k.*

Proof. By the assumption of local slice construction, $\Delta_{(f,g)}$ is irreducible. Since $\operatorname{rank}\Delta_{(f,g)} \leq 2$ by assumption, there exists a coordinate p of $k[\mathbf{x}]$ over k such that $\Delta_{(f,g)}(p) = 0$. Hence, we know by Lemma 12.3 that $\operatorname{pl}\Delta_{(f,g)} = (q)$ for some $q \in k[p] \setminus \{0\}$.

Suppose to the contrary that $\operatorname{rank}\Delta_{(f,g)} = 1$. Then, we have $\ker\Delta_{(f,g)} = \sigma(k[x_2, x_3])$ for some $\sigma \in \operatorname{Aut}(k[\mathbf{x}]/k)$. Since $\ker\Delta_{(f,g)} = k[f, g]$ by (LSC1), it follows that f and g are coordinates of $k[\mathbf{x}]$ over k. Since $\Delta_{(f,h)}(f) = 0$, we have $\operatorname{rank}\Delta_{(f,h)} \leq 2$, a contradiction. Thus, we get $\operatorname{rank}\Delta_{(f,g)} = 2$. By Lemma 12.1, this implies that q does not belong to $k^{\times}$. By (LSC2), there exists $F \in k[f] \setminus \{0\}$ such that $\Delta_{(f,g)}(s) = gF$. Since $D^2(s) = 0$, we see that $D(s)$ belongs to $\operatorname{pl}\Delta_{(f,g)}$. Therefore, q is a factor of gF.

We show that $q \approx g$. First, suppose that q is not divisible by g. Then, q is a factor of F by the irreducibility of g. Since q is an element of $k[p] \setminus k$, we may find $\alpha \in \bar{k}$ such that $p - \alpha$ divides q. Then, $p - \alpha$ is a factor of F. By Lemma 4.1 and the remark following it, $\bar{k}[f]$ is factorially closed in $\bar{k}[\mathbf{x}]$. Hence, $p - \alpha$ belongs to $\bar{k}[f]$. On the other hand, p is a coordinate of $\bar{k}[\mathbf{x}]$ over $\bar{k}$, since a coordinate of $k[\mathbf{x}]$ over k is necessarily a coordinate of $\bar{k}[\mathbf{x}]$ over $\bar{k}$. Hence, $p - \alpha$ is a coordinate of $\bar{k}[\mathbf{x}]$ over $\bar{k}$. In particular, $p - \alpha$ is an irreducible element of $\bar{k}[\mathbf{x}]$. Thus, $p - \alpha$ must be a linear polynomial in f over $\bar{k}$. Since p and f are elements of $k[\mathbf{x}]$, this implies that p is a linear polynomial in f over k. Hence, f is a coordinate of $k[\mathbf{x}]$ over k. Thus, we get $\operatorname{rank}\Delta_{(f,h)} \leq 2$, a contradiction. Therefore, q is divisible by g. Next, suppose that $q \not\approx g$. Then, $q' := g^{-1}q$ is a factor of F. Hence, q' belongs to $k[f] \setminus k$ by the factorially closedness of $k[f]$ in $k[\mathbf{x}]$. Since f and g are algebraically independent over k, it follows that q' and g are algebraically independent over k. This contradicts that $q'g = q$ belongs to $k[p]$. Thus,

we get $q \approx g$, proving that $\mathrm{pl}\,\Delta_{(f,g)} = (g)$. Consequently, g belongs to $k[p]$. Since g is irreducible in $\bar{k}[\mathbf{x}]$, this implies that q is a linear polynomial in p over k. Therefore, g is a coordinate of $k[\mathbf{x}]$ over k. $\square$

For example, D_2 is of rank three if $t_0 \geq 3$ and $(t_0, t_1) \neq (3, 1)$ by Theorem 1.3 (iv), and is obtained from the data (f_2, x_1, r) by a local slice construction. In this case, $D_1 = \Delta_{(f_2,x_1)}$ is triangular. It is previously not known whether there exists an example in which $\mathrm{rank}\,\Delta_{(f,h)} = 3$, $\mathrm{rank}\,\Delta_{(f,g)} = 2$ and $\Delta_{(f,g)}$ is not triangularizable. In closing this section, we construct $f, s \in k[\mathbf{x}]$ such that the data (f, x_1, s) yields a rank three locally nilpotent derivation, and $\Delta_{(f,x_1)}$ is not triangularizable.

Define $p_1, p_2, f \in k[\mathbf{x}]$ by

$$p_1 = (x_1 + 1)x_2 - x_1^2 x_3^2, \quad p_2 = x_1 x_3 + (x_1 + 1)p_1^2$$
$$f = (x_1 + 1)^{-1}(p_1 + p_2^2) = x_2 + 2x_1 x_3 p_1^2 + (x_1 + 1)p_1^4.$$

Then, we have $k(x_1)[f, p_2] = k(x_1)[p_1, p_2] = k(x_1)[p_1, x_3] = k(x_1)[x_2, x_3]$. We show that $D := \Delta_{(f,x_1)}$ is locally nilpotent. Since

$$df \wedge dx_1 \wedge dp_2 = (x_1 + 1)^{-1} dp_1 \wedge dx_1 \wedge dp_2 = -x_1 dx_1 \wedge dx_2 \wedge dx_3,$$

we have $D(p_2) = -x_1$. Since $D(f) = D(x_1) = 0$, it follows that D extends to a locally nilpotent derivation of $k(x_1)[x_2, x_3] = k(x_1)[p_2, f]$ over $R := k(x_1)[f]$. Therefore, D is locally nilpotent. We mention that $\ker D$ is contained in R, and hence in $k(x_1, f)$.

Now, following Daigle [**3**, Example 3.5], we show that D is not triangularizable by contradiction. Suppose to the contrary that D is triangularizable. Then, there exists a coordinate p of $k[\mathbf{x}]$ over $k[x_1]$ such that $k(x_1)[f, p] = k(x_1)[x_2, x_3]$ (see [**3**, Corollary 3.4]). Since $k(x_1)[x_2, x_3] = k(x_1)[f, p_2]$, it follows that

$$R[p] = k(x_1)[f, p] = k(x_1)[x_2, x_3] = k(x_1)[f, p_2] = R[p_2].$$

This implies that $p = ap_2 + b$ for some $a \in R^\times = k(x_1)^\times$ and $b \in R$. Hence, we may write

$$a_0(x_1)p = a_1(x_1)p_2 + \sum_{i \geq 0} b_i(x_1) f^i,$$

where $a_i(x_1) \neq 0$ for $i = 0, 1$ and $b_i(x_1)$ for $i \geq 0$ are elements of $k[x_1]$ with no common factor. Since

$$p_1 \equiv x_2, \quad p_2 \equiv x_2^2, \quad f \equiv x_2 + x_2^4 \pmod{x_1 k[\mathbf{x}]}, \tag{14.1}$$

it follows that

$$a_0(0)p = a_1(0)x_2^2 + \sum_{i \geq 0} b_i(0)(x_2 + x_2^4)^i.$$

When $a_0(0) = a_1(0) = 0$, the preceding equality implies that $b_i(0) = 0$ for every $i \geq 0$. Hence, x_1 is a common factor of $a_0(x_1)$, $a_1(x_1)$ and $b_i(x_1)$ for $i \geq 0$, a contradiction. If $a_0(0) \neq 0$ and $a_1(0) = 0$, then we have $p \approx \sum_{i \geq 0} b_i(0)(x_2 + x_2^4)^i$. Hence, p is an element of $k[x_2]$, and is not a linear

polynomial. This contradicts that p is a coordinate of $k[\mathbf{x}]$ over $k[x_1]$. If $a_0(0) = 0$ and $a_1(0) \neq 0$, then we have

$$b_0(0) + b_1(0)(x_2 + x_2^4) + \left(a_1(0) + \sum_{i \geq 2} b_i(0)(1 + x_2^3)^i x_2^{i-2} \right) x_2^2 = 0.$$

This gives that $b_0(0) \equiv 0 \pmod{x_2 k[x_2]}$, and so $b_0(0) = 0$. Hence, we have $b_1(0)x_2 \equiv 0 \pmod{x_2^2 k[x_2]}$, and so $b_1(0) = 0$. Then, it follows that $b_i(0) = 0$ for every $i \geq 2$, and consequently $a_1(0) = 0$. Thus, x_1 is a common factor of $a_0(x_1)$, $a_1(x_1)$ and $b_i(x_1)$ for $i \geq 0$, a contradiction. Therefore, D is not triangularizable. In particular, we have $\operatorname{rank} D \neq 1$.

We show that D is irreducible. Suppose that D is not irreducible. Then, $D(k[\mathbf{x}])$ is contained in $x_1 k[\mathbf{x}]$, since $D(p_2) = -x_1$ as mentioned. Hence, $x_1^{-1}D$ belongs to $\mathrm{LND}_k\, k[\mathbf{x}]$. Since $(x_1^{-1}D)(-p_2) = 1$, we get $\operatorname{rank} D = \operatorname{rank} x_1^{-1}D = 1$ by Lemma 12.1, a contradiction. Therefore, D is irreducible. Since $\ker D$ is contained in $k(f, x_1)$ as mentioned, we conclude that $\ker D = k[x_1, f]$ by the "kernel criterion" (cf. [**9**, Proposition 5.12]). Hence, $D = \Delta_{(f,x_1)}$ satisfies (LSC1) for $(f, g) = (f, x_1)$. Note that $s := p_2 f + x_1^2$ does not belong to $x_1 k[\mathbf{x}]$, since

$$s \equiv p_2 f \equiv x_2^2(x_2 + x_2^4) \not\equiv 0 \pmod{x_1 k[\mathbf{x}]}.$$

Moreover, we have

$$D(s) = D(p_2 f + x_1^2) = D(p_2)f = -x_1 f.$$

Hence, D satisfies (LSC2) for $s = p_2 f + x_1^2$ and $F = -f$.

Now, define $q = (s^2 - f^3)^2 - f^3 s$. Since $s \equiv p_2 f \pmod{x_1 k[\mathbf{x}]}$, we see that q is congruent to

$$(p_2^2 f^2 - f^3)^2 - f^4 p_2 = f^4\big((p_2^2 - f)^2 - p_2\big)$$

modulo $x_1 k[\mathbf{x}]$. By (14.1), this polynomial is congruent to

$$f^4\Big(\big((x_2^2)^2 - (x_2 + x_2^4)\big)^2 - x_2^2\Big) = 0$$

modulo $x_1 k[\mathbf{x}]$. Hence, q belongs to $x_1 k[\mathbf{x}]$. We show that q is an irreducible element of $k[f, s]$. Write $q = f^4((t^2 - f)^2 - t)$, where $t := s/f$. Then, $q' := (t^2 - f)^2 - t$ is a coordinate of $k[f, t]$ over k, since $k[q', t^2 - f] = k[f, t]$. Hence, q' is an irreducible element of $k[f, t]$. Since q' does not belong to $k[f]$, it follows that q' is an irreducible polynomial in t over $k[f]$, and hence over $k(f)$. Thus, q is an irreducible polynomial in t over $k(f)$. Since $t = s/f$, this implies that q is an irreducible polynomial in s over $k(f)$. Because q is a primitive polynomial in s over $k[f]$, we conclude that q is an irreducible element of $k[f, s]$. Put $h = qx_1^{-1}$. Then, it follows from (a) and (b) of Theorem 4.3 that $D' := \Delta_{(f,h)}$ is locally nilpotent and $D'(s) = hf$.

We show that D' is irreducible. Thanks to Proposition 5.1, it suffices to check that f and $D'(p_2)$ have no common factor. Since $\ker D = k[f, x_1]$, we know by Lemma 4.1 that $fk[\mathbf{x}]$ is a prime ideal of $k[\mathbf{x}]$. Hence, it suffices to verify that $D'(p_2) \not\equiv 0 \pmod{fk[\mathbf{x}]}$. Since $x_1 = qh^{-1}$, $D'(f) = D'(h) = 0$ and $D'(s) = hf$, we have

$$D'(x_1) = D'(qh^{-1}) = D'(q)h^{-1} = \frac{\partial q}{\partial s} D'(s)h^{-1} = (4s(s^2 - f^3) - f^3)f$$

by chain rule. On the other hand, we have

$$hf = D'(s) = D'(p_2 f + x_1^2) = D'(p_2)f + 2x_1 D'(x_1).$$

From the two equalities above, it follows that

$$D'(p_2) = h - 2x_1\bigl(4s(s^2 - f^3) - f^3\bigr).$$

Note that $s \equiv x_1^2$, $q \equiv s^4 \equiv x_1^8$ and $h = qx_1^{-1} \equiv x_1^7 \pmod{fk[\mathbf{x}]}$. By Lemma 4.1, x_1 does not belong to $fk[\mathbf{x}]$. Thus, we know that

$$D'(p_2) \equiv h - 8x_1 s^3 \equiv -7x_1^7 \not\equiv 0 \pmod{fk[\mathbf{x}]}.$$

Therefore, D' is irreducible. Consequently, we get $\ker D' = k[f, h]$ by Theorem 4.3 (c).

Since $D'(s) = hf$ belongs to $\operatorname{pl} D'$, there exists a factor g' of hf such that $\operatorname{pl} D' = (g')$. We show that $g' \approx hf$. Since $\ker D' = k[f, h]$, we know by Lemma 4.1 that f and h are irreducible elements of $k[\mathbf{x}]$ with $f \not\approx h$. Hence, it suffices to check that g' is divisible by h and f.

Suppose to the contrary that g' is not divisible by h. Then, $\operatorname{pl} D'$ is not contained in $hk[\mathbf{x}]$. Hence, we have $(s + \ker D') \cap hk[\mathbf{x}] \neq \emptyset$ by the last part of Lemma 12.3. Since $\ker D' = k[f, h]$, it follows that $(s + k[f]) \cap hk[\mathbf{x}] \neq \emptyset$. Hence, $k[f, s] \cap hk[\mathbf{x}]$ contains a linear polynomial in s over $k[f]$. On the other hand, $k[f, s] \cap hk[\mathbf{x}]$ is a principal prime ideal of $k[f, s]$ by Lemma 4.2 (iii). Since $q = x_1 h$ belongs to $k[f, s] \cap hk[\mathbf{x}]$, and is an irreducible element of $k[f, s]$, we know that $k[f, s] \cap hk[\mathbf{x}]$ is generated by q. Because $\deg_s q = 4 > 1$, it follows that $k[f, s] \cap hk[\mathbf{x}]$ contains no linear polynomial in s over $k[f]$, a contradiction. Therefore, g' is divisible by h.

Suppose to the contrary that g' is not divisible by f. Then, $\operatorname{pl} D'$ is not contained in $fk[\mathbf{x}]$. Hence, we have $(s + \ker D') \cap fk[\mathbf{x}] \neq \emptyset$ by Lemma 12.3. Since $\ker D' = k[f, h]$, we get $(s + k[h]) \cap fk[\mathbf{x}] \neq \emptyset$. Hence, $k[h, s] \cap fk[\mathbf{x}]$ contains a linear polynomial in s over $k[h]$. On the other hand, $k[h, s] \cap fk[\mathbf{x}]$ is a principal prime ideal of $k[h, s]$ by Lemma 4.2 (iii). Since $s^7 - h^2$ is an irreducible element of $k[h, s]$ with $s^7 - h^2 \equiv 0 \pmod{fk[\mathbf{x}]}$, it follows that $k[h, s] \cap fk[\mathbf{x}]$ is generated by $s^7 - h^2$. This contradicts that $k[h, s] \cap fk[\mathbf{x}]$ contains a linear polynomial in s over $k[f]$. Therefore, g' is divisible by f. This proves that $g' \approx hf$. Because f and h are algebraically independent over k, we conclude that $\operatorname{rank} D' = 3$ by Proposition 12.4.

Conclusion

In closing this monograph, we list some problems, questions and conjectures. We assume that $n = 3$, and D is a nonzero element of $\mathrm{LND}_k\, k[\mathbf{x}]$ unless otherwise stated.

Conjecture A. If $\exp D$ is tame, then D is tamely triangularizable.

We claim that Conjecture A is equivalent to the following conjecture.

Conjecture B. If $\exp D$ is tame, then D kills a tame coordinate of $k[\mathbf{x}]$ over k.

In fact, if D is tamely triangularizable, then D kills a tame coordinate of $k[\mathbf{x}]$ over k, since a triangular derivation always kills a tame coordinate. Conversely, Conjecture B implies Conjecture A by virtue of Theorem 1.3 (i).

By the remark after Problem 5, Conjecture A implies the following conjecture. This conjecture is true if D kills a tame coordinate of $k[\mathbf{x}]$ over k.

Conjecture C. If $\exp fD$ is tame for some $f \in \ker D \setminus \{0\}$, then $\exp D$ is tame.

Conjecture B immediately implies the following conjecture.

Conjecture D. If $\operatorname{rank} D = 3$, then $\exp D$ is wild.

Next, we discuss totally (quasi-totally, exponentially) wildness of coordinates of $k[\mathbf{x}]$ over k.

Problem E. For $D \in \mathrm{LND}_k\, k[\mathbf{x}]$ with $\operatorname{rank} D = 3$, study totally (quasi-totally, exponentially) wildness of $(\exp D)(x_i)$ for $i = 1, 2, 3$.

As we have seen in Chapter 6, it is very hard to prove that a coordinate is totally wild or quasi-totally wild.

Problem F. Find simple criteria for totally (quasi-totally, exponentially) wildness of coordinates of $k[\mathbf{x}]$ over k.

The following question is not answered even for Nagata's automorphism.

Question G. Assume that $\sigma \in \mathrm{Aut}(k[\mathbf{x}]/k)$ is wild. Is one of $\sigma(x_1)$, $\sigma(x_2)$ and $\sigma(x_3)$ always quasi-totally wild?

As remarked after Definition 0.1, "quasi-totally wild" implies "exponentially wild". In view of Corollary 1.5 (i), we may ask the following question.

Question H. Is every exponentially wild coordinate of $k[\mathbf{x}]$ over k quasi-totally wild?

Recall that there exists a wild coordinate of $k[\mathbf{x}]$ over k which is not exponentially wild.

Conjecture I. Let f be a wild coordinate of $k[\mathbf{x}]$ over k which is not exponentially wild. Then, there always exists $D \in \mathrm{LND}_k\, k[\mathbf{x}]$ with $\operatorname{rank} D = 1$ such that $D(f) = 0$ and D is tamely triangularizable.

For example, let ψ be Nagata's automorphism defined in (0.2). Then, $\psi(x_2)$ is wild due to Umirbaev-Yu [**29**], but is not exponentially wild as mentioned. In fact, $\psi(x_2)$ is killed by $D \in \mathrm{LND}_k\, k[\mathbf{x}]$ defined in (1.3), which is triangular if x_1 and x_3 are interchanged. Since D kills $x_3 = \psi(x_3)$ as well as $\psi(x_2)$, we know that $\operatorname{rank} D = 1$. Hence, the conjecture is true in this case.

We show that Conjectures A implies Conjecture I. Since f is not exponentially wild, there exists $D \in \mathrm{LND}_k\, k[\mathbf{x}] \setminus \{0\}$ such that $D(f) = 0$ and $\exp D$ is tame. Then, D is tamely triangularizable by Conjecture A, and hence kills a tame coordinate g of $k[\mathbf{x}]$ over k. Since f is wild and g is tame, we know that f is not a linear polynomial in g over k. Hence, we have $k[f] \neq k[g]$. Therefore, we conclude that $\operatorname{rank} D = 1$ from the following lemma.

Lemma 14.2. *Let f and g be coordinates of $k[\mathbf{x}]$ over k such that $k[f] \neq k[g]$. If $D(f) = D(g) = 0$ for $D \in \mathrm{LND}_k\, k[\mathbf{x}] \setminus \{0\}$, then we have* $\operatorname{rank} D = 1$.

Proof. First, we show that f and g are algebraically independent over k. Suppose to the contrary that f and g are algebraically dependent over k. Then, f is algebraic over $k(g)$. Since g is a coordinate of $k[\mathbf{x}]$ over k, we see that $k(g)$ is algebraically closed in $k(\mathbf{x})$, and $k(g) \cap k[\mathbf{x}] = k[g]$. Hence, f belongs to $k[g]$. Similarly, g belongs to $k[f]$. Thus, we get $k[f] = k[g]$, a contradiction. Therefore, f and g are algebraically independent over k.

Now, we prove that $\operatorname{rank} D = 1$. Without loss of generality, we may assume that D is irreducible. Then, we have $\operatorname{pl} D = (p) = (q)$ for some $p \in k[f] \setminus \{0\}$ and $q \in k[g] \setminus \{0\}$ by Proposition 12.3. Since $(\ker D)^\times = k^\times$, it follows that $p = cq$ for some $c \in k^\times$. Because f and g are algebraically independent over k, this implies that p and q belong to $k^\times$. Thus, we get $\operatorname{pl} D = \ker D$. Therefore, we know by Lemma 12.1 that $\operatorname{rank} D = 1$. $\square$

For any $n \geq 3$, it is widely believed that every $\sigma \in \operatorname{Aut}(k[\mathbf{x}]/k)$ is *stably tame*, i.e., the natural extension of σ to an element of

$$\operatorname{Aut}(k[x_1, \ldots, x_r]/k[x_{n+1}, \ldots, x_r])$$

belongs to $\mathrm{T}(k, \{x_1, \ldots, x_r\})$ for some $r \geq n$ (Stable Tameness Conjecture). Assume that $n = 3$. Then, every element of $\operatorname{Aut}(k[\mathbf{x}]/k[x_3])$ is stably tame due to Berson-van den Essen-Wright [**2**] (see also [**28**]). Hence, $\exp D$ is stably tame for each $D \in \mathrm{LND}_k\, k[\mathbf{x}]$ with $D(x_3) = 0$.

The following question is open.

Question J. Is $\exp D$ stably tame for $D \in \mathrm{LND}_k\, k[\mathbf{x}]$ with $\operatorname{rank} D = 3$?

We mention that some $D \in \mathrm{LND}_k\, k[\mathbf{x}]$ with $\operatorname{rank} D = 3$ can be extended to elements of $\mathrm{LND}_k\, k[x_1, \ldots, r]$ with $\operatorname{rank} D = r$ for each $r \geq 4$ in a very simple manner (cf. [**8**, Section 3]). In view of this fact, we may ask the following question, in contrast with Conjecture D.

Question K. Assume that $n \geq 4$. Does there exist $D \in \mathrm{LND}_k\, k[\mathbf{x}]$ such that $\operatorname{rank} D = n$ and $\exp D$ is tame?

For each subgroup G of $\mathrm{Aut}(k[\mathbf{x}]/k)$, we define G^* to be the normal subgroup of $\mathrm{Aut}(k[\mathbf{x}]/k)$ generated by

$$\bigcup_{\sigma \in \mathrm{Aut}(k[\mathbf{x}]/k)} \sigma^{-1} \circ G \circ \sigma.$$

We say that $\phi \in \mathrm{Aut}(k[\mathbf{x}]/k)$ is *absolutely wild* if ϕ does not belong to $\mathrm{T}(k, \mathbf{x})^*$. We note that the wild automorphism defined as in (0.4) is not absolutely wild. We do not know whether Nagata's automorphism is absolutely wild.

The following is an "absolute" version of the Tame Generators Problem.

Problem L. Decide whether $\mathrm{T}(k, \mathbf{x})^*$ equal to $\mathrm{Aut}(k[\mathbf{x}]/k)$.

For each $\alpha \in k^\times$, define $\psi_\alpha \in \mathrm{Aut}(k[\mathbf{x}]/A_1)$ by $\psi_\alpha(x_1) = \alpha x_1$, where A_i is as in (0.1) for each $i \in \{1, 2, 3\}$. Then,

$$\iota : k^\times \ni \alpha \mapsto \psi_\alpha \in \mathrm{Aut}(k[\mathbf{x}]/k)$$

is an injective homomorphism of groups. We set $G_m(k) = \iota(k^\times)$.

Proposition 14.3. *We have $G_m(k)^* = \mathrm{T}(k, \mathbf{x})^*$.*

Proof. Since $\mathrm{T}(k, \mathbf{x}) = \mathrm{E}(k, \mathbf{x})$, it suffices to show that $G_m(k)^*$ contains $\mathrm{Aut}(k[\mathbf{x}]/A_i)$ for $i = 1, 2, 3$. For $i = 2, 3$, define $\tau_i \in \mathrm{Aut}(k[\mathbf{x}]/k)$ by $\tau_i(x_1) = x_i$, $\tau_i(x_i) = x_1$ and $\tau_i(x_j) = x_j$, where $j \in \{2, 3\}$ with $j \neq i$. Then, we have $\tau_i^{-1} \circ \mathrm{Aut}(k[\mathbf{x}]/A_1) \circ \tau_i = \mathrm{Aut}(k[\mathbf{x}]/A_i)$. So we show that $\mathrm{Aut}(k[\mathbf{x}]/A_1)$ is contained in $G_m(k)^*$. Take any $\phi \in \mathrm{Aut}(k[\mathbf{x}]/A_1)$. Then, we have $\phi(x_1) = \alpha x_1 + h$ for some $\alpha \in k^\times$ and $h \in A_1$. Define $\sigma \in G_m(k)^*$ by $\sigma(x_1 + h) = 2(x_1 + h)$ and $\sigma(x_i) = x_i$ for $i = 2, 3$. Then, we have $\sigma(x_1) = 2x_1 + h$, and so $(\psi_{\alpha/2} \circ \sigma)(x_1) = \alpha x_1 + h$. Since $(\psi_{\alpha/2} \circ \sigma)(x_i) = x_i$ for $i = 2, 3$, we conclude that $\psi_{\alpha/2} \circ \sigma = \phi$. Hence, ϕ belongs to $G_m(k)^*$. Thus, $\mathrm{Aut}(k[\mathbf{x}]/A_1)$ is contained in $G_m(k)^*$. Therefore, we get $G_m(k)^* = \mathrm{T}(k, \mathbf{x})^*$. $\square$

Note that

$$\pi : \mathrm{Aut}(k[\mathbf{x}]/k) \ni \phi \mapsto \det J\phi \in k^\times$$

is a homomorphism of groups such that $\pi \circ \iota = \mathrm{id}_{k^\times}$. Hence, $\mathrm{Aut}(k[\mathbf{x}]/k)$ is a semidirect product of $\ker \pi$ and $G_m(k)$. The *Exponential Generators Conjecture* says that $\ker \pi$ is generated by $\exp D$ for $D \in \mathrm{LND}_k\, k[\mathbf{x}]$. For $\emptyset \neq I \subset \{1, 2, 3\}$, we define $\mathrm{Exp}_k^I\, k[\mathbf{x}]$ to be the subgroup of $\mathrm{Aut}(k[\mathbf{x}]/k)$ generated by $\exp D$ for $D \in \mathrm{LND}_k\, k[\mathbf{x}]$ with $\operatorname{rank} D = r$ for $r \in I$. Then, $\mathrm{Exp}_k^I\, k[\mathbf{x}]$ is a normal subgroup of $\mathrm{Aut}(k[\mathbf{x}]/k)$ for each $\emptyset \neq I \subset \{1, 2, 3\}$. However, we know nothing about the structure of $\mathrm{Exp}_k^I\, k[\mathbf{x}]$.

We would like to conclude this monograph with the following remark. When $n \geq 4$, there exist complicated tame automorphisms. Indeed, some wild automorphisms in three variables extend to tame automorphisms in four or more variables by the stable tameness. When $n = 3$, in contrast, the Shestakov-Umirbaev theory suggests that the tame automorphisms are simple. Indeed, the tame automorphisms in three variables can be completely controlled by elementary reductions and some types of reductions. We believe that no tame automorphism in three variables is beyond imagination.

In other words, automorphisms in three variables are *usually* wild unless they are *obviously* tame.

Bibliography

[1] H. Bass, *A nontriangular action of* $\mathbf{G}_a$ *on* $\mathbf{A}^3$, J. Pure Appl. Algebra **33** (1984), 1–5.

[2] J. Berson, A. van den Essen, and D. Wright, Stable tameness of two-dimensional polynomial automorphisms over a regular ring, arXiv:0707.3151v9 [math.AC].

[3] D. Daigle, A necessary and sufficient condition for triangulability of derivations of $k[X, Y, Z]$, J. Pure Appl. Algebra **113** (1996), 297–305.

[4] D. Daigle, D. Daigle, Homogeneous locally nilpotent derivations of $k[X, Y, Z]$, J. Pure Appl. Algebra **128** (1998), 109–132.

[5] D. Daigle and S. Kaliman, *A note on locally nilpotent derivations and variables of* $k[X, Y, Z]$, Canad. Math. Bull. **52** (2009), 535–543.

[6] A. van den Essen, Polynomial automorphisms and the Jacobian conjecture, Progress in Mathematics, Vol. 190, Birkhäuser, Basel, Boston, Berlin, 2000.

[7] G. Freudenburg, Local slice constructions in $k[X, Y, Z]$, Osaka J. Math. **34** (1997), 757–767.

[8] G. Freudenburg, Actions of $\mathbf{G}_a$ on $\mathbf{A}^3$ defined by homogeneous derivations, J. Pure Appl. Algebra **126** (1998), 169–181.

[9] G. Freudenburg, Algebraic theory of locally nilpotent derivations, Encyclopaedia Math. Sci., 136, Springer, Berlin, 2006.

[10] J.-P. Furter, On the variety of automorphisms of the affine plane, J. Algebra **195** (1997), 604–623.

[11] O. Hadas and L. Makar-Limanov, Newton polytopes of constants of locally nilpotent derivations, Comm. Algebra **28** (2000), 3667–3678.

[12] H. Jung, Über ganze birationale Transformationen der Ebene, J. Reine Angew. Math. **184** (1942), 161–174.

[13] M. Karaś and J. Zygadło, On multidegrees of tame and wild automorphisms of $\mathbf{C}^3$, to appear in J. Pure Appl. Algebra.

[14] W. van der Kulk, On polynomial rings in two variables, Nieuw Arch. Wisk. (3) **1** (1953), 33–41.

[15] S. Kuroda, A generalization of the Shestakov-Umirbaev inequality, J. Math. Soc. Japan **60** (2008), 495–510.

[16] S. Kuroda, Shestakov-Umirbaev reductions and Nagata's conjecture on a polynomial automorphism, Tohoku Math. J. **62** (2010), 75–115.

[17] S. Kuroda, Wildness of polynomial automorphisms: Applications of the Shestakov-Umirbaev theory and its generalization, in *Higher Dimensional Algebraic Geometry*, 103–120, RIMS Kokyuroku Bessatsu, B24 Res. Inst. Math. Sci. (RIMS), Kyoto, 2011.

[18] S. Lang, Algebra, revised third edition, Graduate Texts in Mathematics, 211, Springer, New York, 2002.

[19] H. Matsumura, Commutative ring theory, translated from the Japanese by M. Reid, second edition, Cambridge Studies in Advanced Mathematics, 8, Cambridge Univ. Press, Cambridge, 1989.

[20] M. Miyanishi, *Curves on rational and unirational surfaces*, Tata Institute of Fundamental Research Lectures on Mathematics and Physics, 60, Tata Inst. Fund. Res., Bombay, 1978.

[21] M. Miyanishi, Normal affine subalgebras of a polynomial ring, Algebraic and Topological Theories—to the memory of Dr. Takehiko Miyata (Tokyo), Kinokuniya, 1985, pp. 37–51.

[22] M. Nagata, On Automorphism Group of $k[x, y]$, Lectures in Mathematics, Department of Mathematics, Kyoto University, Vol. 5, Kinokuniya Book-Store Co. Ltd., Tokyo, 1972.

[23] A. Nowicki, Polynomial derivations and their rings of constants, Uniwersytet Mikolaja Kopernika, Torun, 1994.

[24] V. L. Popov, On actions of $\mathbf{G}_a$ on $\mathbf{A}^n$, in *Algebraic groups Utrecht 1986*, 237–242, Lecture Notes in Math., 1271, Springer, Berlin.

[25] R. Rentschler, Opérations du groupe additif sur le plan affine, C. R. Acad. Sci. Paris Sér. A-B **267** (1968), 384–387.

[26] I. Shestakov and U. Umirbaev, Poisson brackets and two-generated subalgebras of rings of polynomials, J. Amer. Math. Soc. **17** (2004), 181–196.

[27] I. Shestakov and U. Umirbaev, The tame and the wild automorphisms of polynomial rings in three variables, J. Amer. Math. Soc. **17** (2004), 197–227.

[28] M. K. Smith, Stably tame automorphisms, J. Pure Appl. Algebra **58** (1989), 209–212.

[29] U. Umirbaev and J.-T. Yu, The strong Nagata conjecture, Proc. Natl. Acad. Sci. USA **101** (2004), 4352–4355.